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STABILITY OF SHELLS OF REVOLUTION: GENERAL THEORY AND APPLICATION TO THE TORUS

by

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FOREWORD

This report was prepared as a dissertation submitted by Lawrence H. Sobel to the Division of Engineering Mechanics, Stanford University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ILLUSTRATIONS

NOTATION

a	meridional radius of curvature for a toroidal shell with circular cross section (see Fig. VI – 1)
b	distance between the center of the cross section and the axis of a toroidal shell (see Fig. VI – 1)
C_m	$= \cos m\psi$
D	$= \dfrac{Eh}{1 - \nu^2}$, extensional stiffness
E	Young's modulus
h	thickness of shell
k	$= 1/12\ (h/a)^2$
K	$= \dfrac{Eh^3}{12(1 - \nu^2)}$, bending stiffness
m	Fourier index
M	number of terms used in series expansions for the displacement components
n	number of circumferential waves in buckle pattern
$N_\phi, N_\theta, N_{\theta\phi}$, $N_{\phi\theta}, Q_\phi, Q_\theta$, M_ϕ, M_θ, $M_{\phi\theta}, M_{\theta\phi}$	incremental stress resultants (see Fig. III – 2)
$N_{\phi 0}, N_{\theta 0}$, $N_{\phi\theta 0}, N_{\theta\phi 0}$	prebuckling stress resultants (see Fig. IV – 1)
p	external pressure loading for a toroidal shell

p_ϕ, p_θ, p_z	components of the applied loading per unit area of the shell's middle surface (see Fig. IV – 1)
r	radius of a parallel circle (see Fig. III – 1)
r_1, r_2	meridional and circumferential radii of curvature (see Fig. III – 1)
S_m	$= \sin m\psi$
t	time
$u_n(\phi), v_n(\phi), w_n(\phi)$	displacement functions
u, v, w	circumferential, meridional, and radial displacement components of a point in the shell's middle surface (see Fig. III – 3)
$\tilde{U}_m, \tilde{V}_m, \tilde{W}_m$	Fourier coefficients [see Eq. (VI – 35)]
U_m, V_m, W_m	Fourier coefficients [see Eqs. (VI – 20)]
x_1	$= \sin \varphi$
y_1, y_2	$= \cos \phi$, $\cos 2\varphi$
z	radial coordinate (see Fig. III – 2)
α	$= b/a$
δ_{mr}	Kronecker delta [see Eq. (VI-25)]
δ_{ph}	defined in Eq. (IV – 4)
ϵ_{mr}	defined in Eq. (VI – 27)
$\epsilon_\phi, \epsilon_\theta, \gamma_{\phi\theta}$	strains at a distance z from the middle surface
$\bar{\epsilon}_\phi, \bar{\epsilon}_\theta, \bar{\gamma}_{\phi\theta}$	middle surface strains
$\kappa_\phi, \kappa_\theta, \kappa_{\phi\theta}$	curvature changes
λ	$= pa/Eh$, nondimensional pressure parameter for a toroidal shell
θ, φ	circumferential and meridional coordinates of a point on the shell's middle surface (see Fig. III – 1)

ν	Poisson's ratio
μ	mass per unit area of the shell's middle surface
ψ	$= \varphi - \frac{\pi}{2}$, meridional coordinate for a toroidal shell (see Fig. VI – 1)
Ω	$= \frac{\mu a^2 \omega^2}{Eh}$, nondimensional frequency parameter for a toroidal shell
ω	$\begin{cases} \frac{1}{\lambda} \text{ for stability of a toroidal shell} \\ \quad\text{or} \\ \text{circular frequency for vibration of a toroidal shell} \end{cases}$
ω_θ, ω_φ	angles of rotation of the normal to the shell's middle surface (see Fig. III – 3)
ω_z, ω_{z1}, ω_{z2}	angles of rotation around the normal to the shell's middle surface [see Fig. III – 3 and Eqs. (III-10) and (III-11)]
$(\ )'$	$= \frac{\partial(\ )}{\partial\theta}$
$(\dot{\ })$ or $(\ )^{\cdot}$	$= \frac{\partial(\ )}{\partial\phi} = \frac{\partial(\ )}{\partial\psi}$

I

INTRODUCTION

The first solution in the field of buckling of thin shells was given in Lorenz's paper on axially symmetric buckling of axially compressed cylinders (Ref. 1). In 1932, Flügge (Ref. 2) developed a general theory for buckling of cylinders and presented numerical results for simply supported cylinders under various loading conditions. For the case of the uniformly loaded cylinder, Flügge made the usual assumption that the prebuckling stresses could be approximated by a homogeneous membrane state of stress. Therefore, the stability equations contain constant coefficients, and exact solutions can readily be obtained for isotropic or orthotropic cylinders with arbitrary boundary conditions. However, when the applied loading is nonuniform the differential equations governing the stability of the cylinder will have variable coefficients. Consequently it becomes considerably more difficult to obtain solutions; for example, see Flügge's analysis of a nonuniformly compressed cylinder (Ref. 2).

For shells other than cylindrical, the stability equations contain variable coefficients. Most of the work done on noncylindrical shells has been devoted to spheres (Refs. 3 through 5) and cones (Refs. 6 and 7). Relatively little attention has been devoted to shells with variable Gaussian curvature. Mushtari and Galimov (Ref. 8), using shallow shell equations, presented a simple formula for the critical normal pressure of an ellipsoidal shell. Their analysis, however, appears to be greatly oversimplified. Machnig (Refs. 9 and 10) investigated the stability of a torus subject to uniform external pressure. In his first paper (Ref. 9), Machnig studied both axially symmetric and asymmetric buckling modes and concluded that the former gives the smallest critical pressure. A perturbation technique was used to solve a system of partial differential equations governing

the asymmetric buckling mode. Of course, it should be possible to separate the space variables in the stability equations for a complete shell of revolution subject to axially symmetric loads and thereby obtain a system of ordinary differential equations. Apparently, Machnig had to contend with partial differential equations instead of ordinary differential equations since it appears that it is not possible to separate the space variables in his equations for the asymmetric buckling mode. In his more recent paper (Ref. 10), Machnig considers only the axially symmetric mode. A review of that paper was given by Koiter (Ref. 11). The reviewer indicates that asymmetric buckling modes may well result in smaller buckling pressures for some values of the shell's geometric parameters and that the power series expansion must break down for toroidal shells with small values of b/a (see Fig. VI-1* for notation). In spite of these critical remarks, Koiter concludes that Machnig's paper must be regarded as a significant first step in the solution for the buckling of a torus.

In this work, the stability of a general shell of revolution subject to arbitrary loads will be investigated. First, the elastic law which relates the incremental stress resultants to the incremental displacement components will be derived. This law, derived from elementary considerations, turns out to be the same as the elastic law derived by Reissner (Ref. 12) who used the methods of differential geometry. Next, the equations of equilibrium will be applied to a differential element of the deformed shell. The resulting partial differential equations are linear and homogeneous in the incremental quantities, and the specification of linear and homogeneous boundary conditions results in an eigenvalue problem.

*Combined Roman-Arabic numbers designate cross-chapter references of figures or equations.

Thereafter, only complete shells of revolution under axially symmetric loads will be considered. This means that the coefficients in all equations are independent of the circumferential coordinate θ and that all incremental quantities are periodic functions of θ (see Fig. III-1 for notation). Therefore, it is possible to express all incremental quantities as Fourier series in θ and replace the partial differential equations by ordinary differential equations. Then, the stress resultants will be eliminated from the equilibrium equations, and three ordinary differential equations for the three incremental displacement components will be obtained.

Next, by specialization of these equations, the ones governing the stability of a toroidal shell subject to a uniform external pressure are obtained. It turns out, as might be expected, that two types of asymmetric buckling modes exist: one which is symmetric with respect to the plane $\psi = 0, \pi$ (see Fig. VI-1) and one which is antimetric with respect to this plane. This will be evident when series expansions are used for the incremental displacement components; as a result, two uncoupled systems of linear homogeneous algebraic equations are obtained for the free constants in the series expansions. A matrix iteration technique is employed to obtain the lowest eigenvalue and the corresponding eigenvector for each system. After the eigenvector has been computed, the mode shapes are given. The matrix iteration method is programmed for the IBM 7094 digital computer and numerical results are presented in the form of design curves which give nondimensional buckling pressures for a wide range of the shell's geometric parameters. These results are compared with available test results as well as with Machnig's results.

Finally, equations for the free vibrations of a pre-stressed shell of revolution will be derived and then specialized for the toroidal shell subject to external or internal pressure.

II

FUNDAMENTAL CONCEPTS AND ASSUMPTIONS

1. Concept of Buckling

In order to establish basic notions and at the same time to introduce a certain terminology, we will discuss the concept of bifurcation buckling in this section. The ideas expressed here may be found in many works; e.g., Refs. 13 through 18.

Let us consider a conservative mechanical system subjected to applied loads. The distribution and direction of the loads are assumed to be known. The magnitude of the loads is assumed to be characterized by a single non-negative scalar λ, which we call the load parameter. When λ is zero, all the applied loads vanish and the system is undeformed. For other values of λ, the system will deform and develop stresses in order to be in equilibrium under the applied loads. Let the load parameter increase monotonically from its initial zero value. Let us assume that for sufficiently small values of λ, there exists a unique solution to the equilibrium problem for the system. The equilibrium configuration given by this solution is called the basic state or prebuckled state. It may happen that for some value of λ, say $\lambda = \lambda^*$, there exists another solution or equilibrium configuration, called the buckled state, which is infinitesimally close to the basic state. If such a value λ^* exists, then it is called the critical value of the load parameter. The objective of the buckling analysis is to determine λ^*. The existence of two adjacent equilibrium configurations for the same value of the load parameter means that there is a bifurcation of the basic state. The stresses and displacements of the prebuckled state are called the prebuckling stresses and prebuckling displacements. At the critical value

of the load parameter, the differences between the stresses and displacements in the prebuckled and buckled states are called the <u>additional</u> or <u>incremental</u> <u>stresses</u> and <u>displacements</u>.

In the present analysis, the mechanical system is a shell of revolution; the applied loads are the conservative external pressures $p_\theta(\lambda)$, $p_\phi(\lambda)$, $p_z(\lambda)$; the basic state is assumed to be a membrane state of stress; the prebuckling stress resultants are denoted by $N_{\theta 0}$, $N_{\phi 0}$, $N_{\theta\phi 0}$, $N_{\phi\theta 0}$; and the incremental quantities are denoted by u, v, w, N_θ, N_ϕ, $N_{\theta\phi}$, $N_{\phi\theta}$, Q_θ, Q_ϕ, M_θ, M_ϕ, $M_{\phi\theta}$.

In the mathematical formulation of the stability problem, we assume that the shell is in a buckled state and we seek the smallest value of the load parameter for which all the basic equations can be satisfied. The basic equations in shell theory, or in any branch of solid mechanics, are the equations of equilibrium, the equations for kinematics, and the constitutive equations (Hooke's law for a linearly elastic material). Before presenting these equations, we list in the next section the assumptions and limitations used in the analysis.

2. <u>Assumptions and Limitations</u>

(a) The shell is made from an isotropic and homogeneous material which obeys Hooke's law (linearly elastic material).

(b) The thickness of the shell is constant.

(c) The thickness of the shell is small in comparison with the radii of curvature of the middle surface (thin shell approximation).

(d) Kirchhoff-Love hypothesis

(i) A straight line normal to the middle surface before deformation remains straight and normal to the middle surface after deformation and retains its original length.

(ii) The normal stresses acting on surfaces parallel to the middle surface are small in comparison with other stresses and may be neglected in the stress-strain relations.

(e) All incremental quantities are infinitesimal.

(f) The incremental strains are small in comparison with the incremental rotations.

(g) The basic state may be approximated by a membrane state of stress.

III

THE ELASTIC LAW

The relations between the incremental stress resultants and the incremental displacement components of the middle surface will be presented in this chapter. These relations, which represent the elastic law for the shell, are obtained through combination of the equations of kinematics and the constitutive equations. The elastic law for the shell buckling problem considered here must be identical to that for the problem of infinitesimal bending of a shell of revolution subject to external loads. Therefore, we could use the elastic law derived by Flügge (Ref. 13) in his analysis of the linear bending of a shell of revolution under external loads. However, instead of proceeding directly to Flügge's final results, we will closely follow his derivation of the elastic law in order to see what assumptions are made and at what stage they are made. The elastic law derived here will differ from the one derived by Flügge because we will deviate somewhat from his derivation.

1. Geometry of a Shell of Revolution

The middle surface of a shell of revolution is shown in Fig. 1a. The equation defining the meridian, $y = y(r)$ or $y = y(\phi)$, is assumed to be given. The lines of principal curvature are the meridians (θ = const.) and parallel circles (ϕ = const.). A point P on the middle surface of the shell is determined by the intersection of a meridian and a parallel circle. Accordingly, we use the angles θ and ϕ as the curvilinear coordinates of the middle surface of the shell.

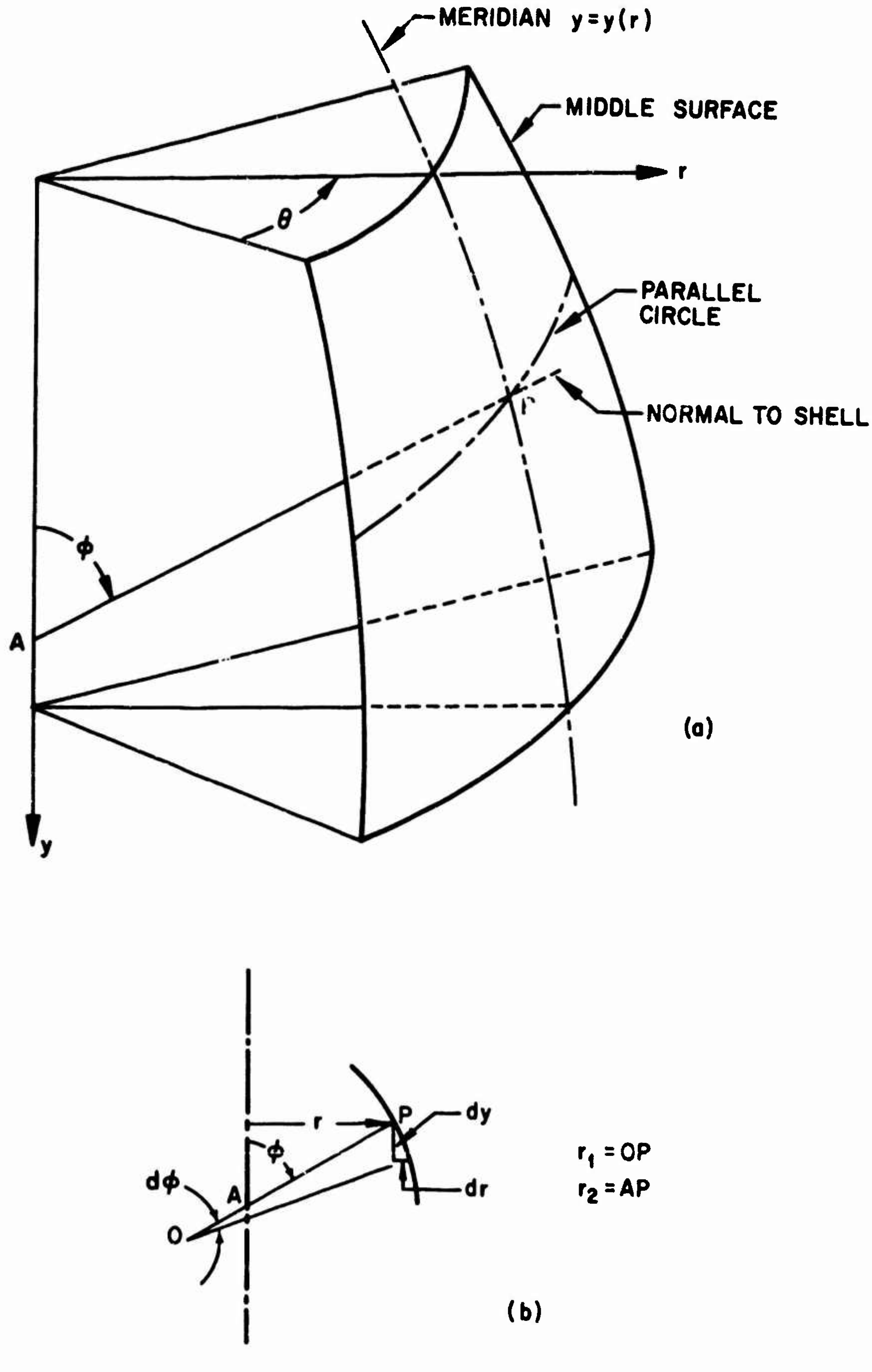


Fig. 1 Notation for a Shell of Revolution

The principal radii of curvature r_1 and r_2 are shown in Fig. 1b. From this figure we can obtain the following useful relations

$$r = r_2 \sin \varphi \tag{1}$$

$$\frac{dr}{d\varphi} = r_1 \cos \varphi \ . \tag{2}$$

The following notation will be used for the derivatives with respect to the independent variables θ and φ

$$\frac{\partial}{\partial \theta}(\ ) = (\ )' \tag{3}$$

$$\frac{\partial}{\partial \phi}(\ ) = (\ )^{\bullet} \text{ or } (\dot{\ }) \tag{4}$$

2. <u>Stress Resultants</u>

Integration of the stress components through the thickness h yields expressions for the stress resultants acting on a unit length of the section θ = const. or φ = const. (see Fig. 2 for notation and sign conventions):

$$N_\varphi = \int_{-h/2}^{h/2} \sigma_\varphi (1 + z/r_2)\, dz \tag{5a}$$

$$N_\theta = \int_{-h/2}^{h/2} \sigma_\theta (1 + z/r_1)\, dz \tag{5b}$$

$$N_{\varphi\theta} = \int_{-h/2}^{h/2} \tau_{\varphi\theta} (1 + z/r_2)\, dz \tag{5c}$$

$$N_{\theta\varphi} = \int_{-h/2}^{h/2} \tau_{\theta\varphi} (1 + z/r_1)\, dz \tag{5d}$$

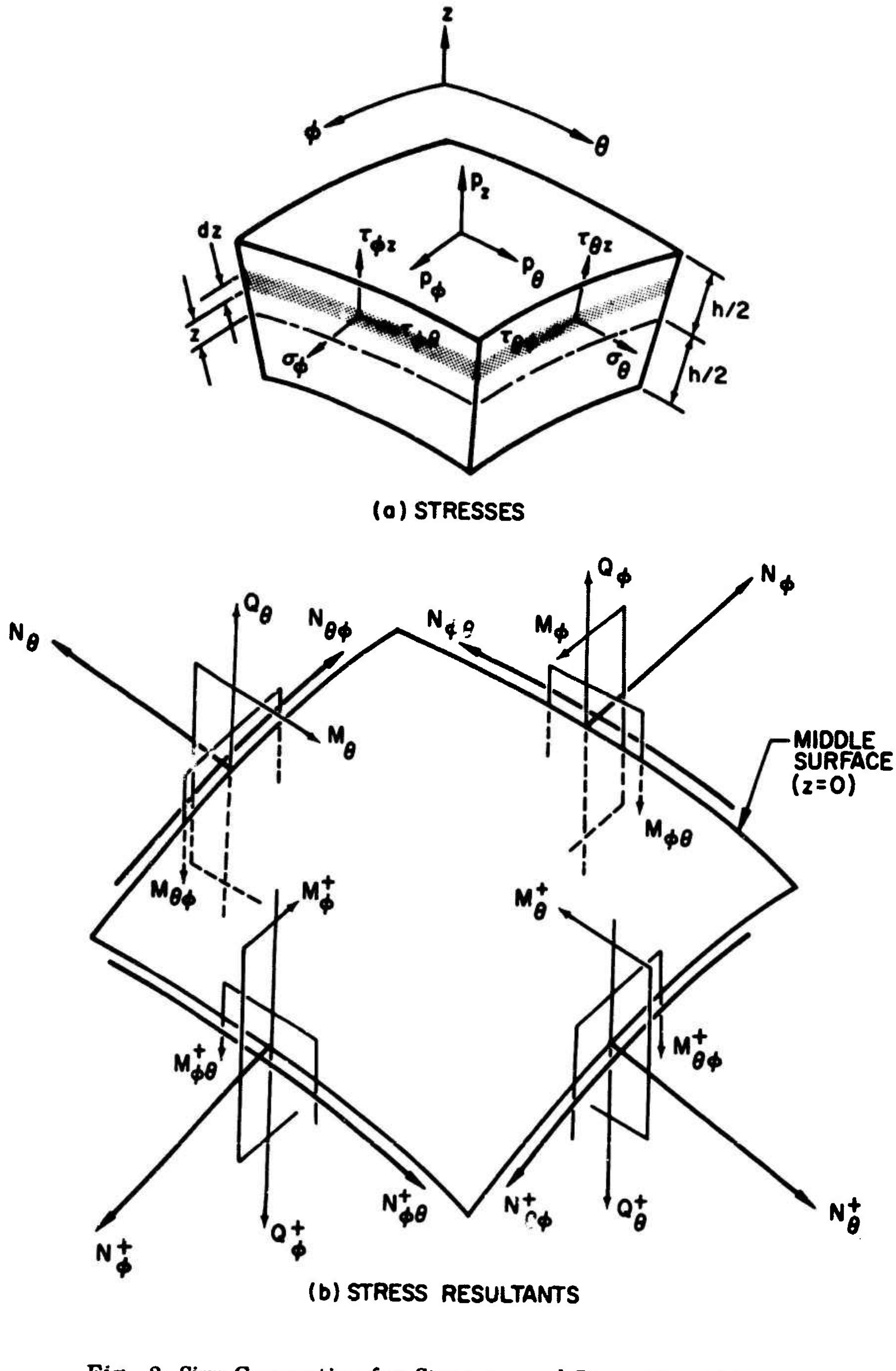


Fig. 2 Sign Convention for Stresses and Stress Resultants

$$Q_\varphi = -\int_{-h/2}^{h/2} \tau_{\varphi z} (1 + z/r_2)\, dz \qquad (5c)$$

$$Q_\theta = -\int_{-h/2}^{h/2} \tau_{\theta z} (1 + z/r_1)\, dz \qquad (5f)$$

$$M_\varphi = -\int_{-h/2}^{h/2} z\sigma_\varphi (1 + z/r_2)\, dz \qquad (5g)$$

$$M_\theta = -\int_{-h/2}^{h/2} z\sigma_\theta (1 + z/r_1)\, dz \qquad (5h)$$

$$M_{\varphi\theta} = -\int_{-h/2}^{h/2} z\tau_{\varphi\theta} (1 + z/r_2)\, dz \qquad (5i)$$

$$M_{\theta\varphi} = -\int_{-h/2}^{h/2} z\tau_{\theta\varphi} (1 + z/r_1)\, dz \qquad (5j)$$

The present analysis is for a thin shell (Assumption II-c). Therefore, $h/r_i \ll 1$ $(i = 1, 2)$ and, in Eqs. (5), we neglect the terms z/r_i in comparison with unity. As a result, $N_{\theta\varphi} = N_{\varphi\theta}$ and $M_{\theta\varphi} = M_{\varphi\theta}$.

3. Constitutive Equations

The Assumption (II-a) that the shell is made from a linearly elastic material enables us to express strains in terms of stresses by means of Hooke's law.

Since normal stresses on surfaces parallel to the middle surface are neglected (Assumption II-d), Hooke's law takes the form

$$\epsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \nu\sigma_\theta) \tag{6a}$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_\varphi) \tag{6b}$$

$$\gamma_{\varphi\theta} = \frac{2(1 + \nu)}{E}\tau_{\phi\theta} \tag{6c}$$

where E is Young's modulus and ν is Poisson's ratio.

Solving Eqs. (6) for the stresses in terms of the strains, we find

$$\sigma_\varphi = \frac{E}{1 - \nu^2}(\epsilon_\varphi + \nu\epsilon_\theta) \tag{7a}$$

$$\sigma_\theta = \frac{E}{1 - \nu^2}(\epsilon_\theta + \nu\epsilon_\varphi) \tag{7b}$$

$$\tau_{\varphi\theta} = \frac{E}{2(1 + \nu)}\gamma_{\varphi\theta} \; . \tag{7c}$$

4. Displacement and Rotation Components

The circumferential, meridional, and radial displacement components of a point P on the middle surface of the shell are denoted by u , v , and w , respectively (see Fig. 3a). The displacement components u and v are taken positive in the direction of increasing θ and φ , respectively, w is positive when it points away from the center of curvature of the meridian.

In the present analysis, the rotation components ω_φ , ω_θ , and ω_z will be used to determine the contributions of prebuckling quantities to equilibrium of a deformed shell element. The angle of rotation of the normal to the middle surface about the tangent to the parallel circle at P is denoted by ω_θ (see Fig. 3b). Now because of the Kirchhoff-Love hypothesis (Assumption II-d), ω_θ can be

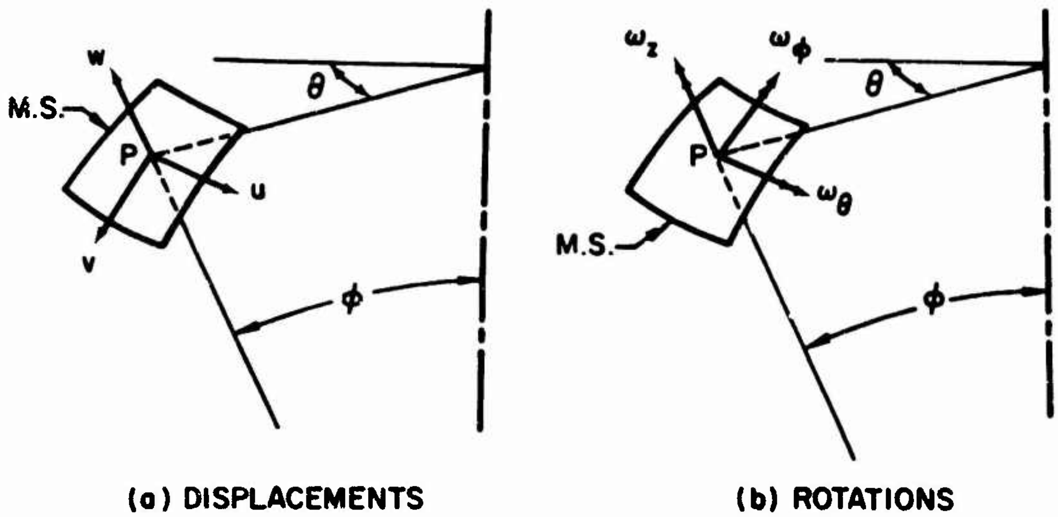


Fig. 3 Displacement and Rotation Components

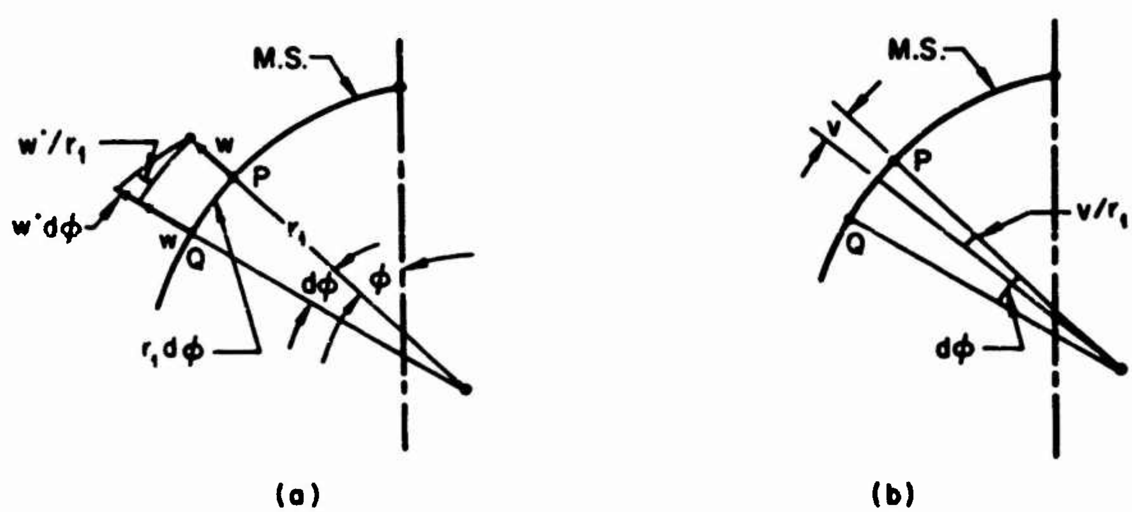


Fig. 4 Rotation Component ω_θ

related to the displacement components of the middle surface. To establish this relationship, let us consider the displacement of two neighboring points P and Q on the same meridional curve of length $r_1 d\phi$ (see Fig. 4). The radial displacement of Q exceeds that of P by an amount $\dot{w} d\phi$ (see Fig. 4a). Consequently, the tangent to the meridian at P rotates through an angle $\dot{w} d\phi / r_1 d\phi$ and, by virtue of the Kirchhoff-Love assumption, the normal at P rotates through the same amount. The displacement v along the meridian causes the normal at P to rotate through an angle v/r_1 (see Fig. 4b). The circumferential displacement u does not contribute to the rotation of the normal about the tangent to the parallel circle at P . Hence, by adding the two contributions to ω_θ in accordance with the sign convention shown in Fig. 3b, we obtain

$$\omega_\theta = -\left(\frac{\dot{w} - v}{r_1}\right) . \tag{8}$$

Proceeding in the same way, we can show that the angle of rotation of the normal about the tangent to the meridian at P is given by

$$\omega_\phi = -\left(\frac{w' - u \sin\phi}{r}\right) , \tag{9}$$

where the positive sense of ω_ϕ is shown in Fig. 3b.

The rotation of the shell element around the normal, denoted by ω_z (see Fig. 3b), is not clearly defined because of the in-surface shear deformation. The angle of rotation ω_{z1} of the tangent to the parallel circle around the normal differs from the angle of rotation ω_{z2} of the tangent to the meridian around the normal. Instead of using an average rotation $\omega_z = \omega_{z1} + \omega_{z2}$, we more

properly use the two separate rotations ω_{z1} and ω_{z2} when we determine the contributions of the prebuckling quantities to the equilibrium equations of a deformed element. We let

$$\omega_z = \begin{cases} \omega_{z1} \\ \omega_{z2} \end{cases} , \tag{10}$$

where it is understood that $\omega_z = \omega_{z1}$ if the prebuckling quantity under consideration acts in the direction of the tangent to the parallel circle, and that $\omega_z = \omega_{z2}$ if the prebuckling quantity acts in the direction of the tangent to the meridian. From Ref. 13, ω_{z1} and ω_{z2} are related to the displacement components of the middle surface by

$$\begin{aligned} \omega_{z1} &= -\frac{v'}{r} \\ \omega_{z2} &= \frac{\dot{u}}{r_1} - \frac{u}{r}\cos\phi \quad . \end{aligned} \tag{11}$$

5. <u>Equations of Kinematics</u>

Let A be a point located at a distance z from the middle surface and let the normal through A intersect the middle surface at the point P . From Eqs. (VI-2) Ref. 13, the strain-displacement equations for point A are

$$\epsilon_\phi = \frac{\dot{v}_A + w_A}{r_1\left(1 + \frac{z}{r_1}\right)} \tag{12a}$$

$$\epsilon_\theta = \frac{u'_A + v_A \cos\phi + w_A \sin\phi}{r\left(1 + \frac{z}{r_2}\right)} \tag{12b}$$

$$\gamma_{\phi\theta} = \frac{\dot{u}_A}{r_1\left(1 + \frac{z}{r_1}\right)} - \frac{u_A \cos\varphi - v'_A}{r\left(1 + \frac{z}{r_2}\right)} \tag{12c}$$

Once again, we invoke the thin-shell assumption and thus neglect z/r_i in Eqs. (12). Hence

$$\epsilon_\varphi = \frac{\dot{v}_A + w_A}{r_1} \tag{13a}$$

$$\epsilon_\theta = \frac{u'_A + v_A \cos\varphi + w_A \sin\varphi}{r} \tag{13b}$$

$$\gamma_{\varphi\theta} = \frac{\dot{u}_A}{r_1} - \frac{u_A \cos\varphi - v'_A}{r} \tag{13c}$$

The right-hand sides of Eqs. (13) contain the displacement components u_A, v_A, and w_A of point A. Now, the Kirchhoff-Love hypothesis enables us to relate u_A, v_A, and w_A to the displacement components u, v, and w of point P on the middle surface and to the rotation components ω_θ and ω_φ. Since ω_θ and ω_φ can be expressed in terms of u, v, and w by means of Eqs. (8) and (9), we see that the adoption of this hypothesis is equivalent to reducing the problem of determining the displacements of the shell to that of determining the displacements of the middle surface. The Kirchhoff-Love hypothesis implies that the displacement components vary linearly through the thickness and that the radial displacement component w_A is independent of z, i.e.,

$$u_A = u + \omega_\varphi z \tag{14a}$$

$$v_A = v + \omega_\theta z \tag{14b}$$

$$w_A = w \quad . \tag{14c}$$

By inserting into Eqs. (14) the expressions for ω_θ and ω_φ given in Eqs. (8)and (9), we obtain the relations between the displacement components of points A and P :

$$u_A = u + (\underline{u \sin\varphi} - w')\frac{z}{r} = u\left(1 + \underline{\frac{z}{r_2}}\right) - w'\frac{z}{r} \tag{15a}$$

$$v_A = v + (\underline{v} - \dot{w})\frac{z}{r_1} = v\left(1 + \underline{\frac{z}{r_1}}\right) - \dot{w}\frac{z}{r_1} \tag{15b}$$

$$w_A = w \quad . \tag{15c}$$

These equations are the same as those derived by Flügge (Ref. 13). The more general elastic law derived in Ref. 13 was simplified for a thin shell by replacing the terms $(1 + z/r_1)$ by unity. In Ref. 13, and in the present analysis, this thin shell approximation was introduced into the equations defining the stress resultants [Eqs. (5)] and into the strain-displacement relations [Eqs. (12)]. Reference 13 applied the thin shell approximation to Eqs. (15) and, as a result, neglected the underlined terms in Eqs. (15). For example, in Eq. (15b), the term $v\frac{z}{r_1}$ was neglected in comparison with v . However, in the present analysis, v/r_1 is interpreted as being a part of the rotation

$$\omega_\theta = -\left(\frac{\dot{w} - v}{r_1}\right)$$

and, as such, is not neglected. Therefore, the underlined terms in Eqs. (15) are not neglected in the present analysis.

6. Elastic Law

From Eqs. (13) and (14), we obtain the strains ϵ_θ , ϵ_φ and $\gamma_{\varphi\theta}$ at a distance z from the middle surface in terms of the displacements u , v , and w at the middle surface and their derivatives:

$$\epsilon_\varphi = \bar{\epsilon}_\varphi - z\kappa_\varphi \tag{16a}$$

$$\epsilon_\theta = \bar{\epsilon}_\theta - z\kappa_\theta \tag{16b}$$

$$\gamma_{\varphi\theta} = \gamma_{\theta\varphi} = \bar{\gamma}_{\varphi\theta} - z(2\kappa_{\varphi\theta}) \ , \tag{16c}$$

where

$$\bar{\epsilon}_\varphi = \frac{\dot{v} + w}{r_1} \tag{17a}$$

$$\bar{\epsilon}_\theta = \frac{u' + v\cos\varphi + w\sin\varphi}{r} \tag{17b}$$

$$\bar{\gamma}_{\varphi\theta} = \frac{\dot{u}}{r_1} - \frac{u\cos\varphi - v'}{r} \tag{17c}$$

$$\kappa_\varphi = -\frac{1}{r_1}\dot{\omega}_\theta \tag{17d}$$

$$\kappa_\theta = -\frac{1}{r}\omega'_\varphi - \frac{1}{r}\omega_\theta\cos\varphi \tag{17e}$$

$$2\kappa_{\varphi\theta} = -\frac{1}{r_1}\dot{\omega}_\varphi - \frac{1}{r}\omega'_\theta + \frac{1}{r}\omega_\theta\cos\varphi \ , \tag{17f}$$

and, from Eqs. (8) and (9),

$$\omega_\theta = -\left(\frac{\dot{w} - v}{r_1}\right)$$

$$\omega_\varphi = -\left(\frac{w' - u\sin\varphi}{r}\right) .$$

The strains ϵ_φ , ϵ_θ , and $\gamma_{\varphi\theta}$ from Eqs. (16) are now entered on the right-hand side of Eqs. (7) to give the stresses σ_φ , σ_θ , and $\tau_{\varphi\theta}$ in terms of u , v , and w . We now introduce these stresses into the integrals [Eqs. (5)], which define the stress resultants, and finally obtain the elastic law for a shell of revolution:

$$N_\varphi = D(\bar{\epsilon}_\varphi + \nu\bar{\epsilon}_\theta) = D\left[\frac{\dot{v} + w}{r_1} + \nu\left(\frac{u' + v\cos\varphi + w\sin\varphi}{r}\right)\right] \tag{18a}$$

$$N_\theta = D(\bar{\epsilon}_\theta + \nu\bar{\epsilon}_\varphi) = D\left[\frac{u' + v\cos\varphi + w\sin\varphi}{r} + \nu\left(\frac{\dot{v} + w}{r_1}\right)\right] \tag{18b}$$

$$N_{\varphi\theta} = \frac{D(1-\nu)}{2}\bar{\gamma}_{\varphi\theta} = \frac{D(1-\nu)}{2}\left[\frac{\dot{u}}{r_1} - \frac{u\cos\varphi - v'}{r}\right] \tag{18c}$$

$$\begin{aligned} M_\varphi = K(\kappa_\varphi + \nu\kappa_\theta) &= K\left[\left(-\frac{1}{r_1}\dot{\omega}_\theta\right) + \nu\left(-\frac{1}{r}\omega_\varphi' - \frac{1}{r}\omega_\theta\cos\varphi\right)\right] \\ &= K\left\{+\frac{1}{r_1}\left(\frac{\dot{w} - v}{r_1}\right)^{\cdot} + \nu\left[\frac{1}{r}\left(\frac{w' - u\sin\varphi}{r}\right)'\right.\right. \\ &\qquad \left.\left. + \frac{1}{r}\left(\frac{\dot{w} - v}{r_1}\right)\cos\varphi\right]\right\} \end{aligned} \tag{18d}$$

$$\begin{aligned} M_\theta = K(\kappa_\theta + \nu\kappa_\varphi) &= K\left[\left(-\frac{1}{r}\omega_\varphi' - \frac{1}{r}\omega_\theta\cos\varphi\right) + \nu\left(-\frac{1}{r_1}\dot{\omega}_\theta\right)\right] \\ &= K\left\{\frac{1}{r}\left(\frac{w' - u\sin\varphi}{r}\right)' + \frac{1}{r}\left(\frac{\dot{w} - v}{r_1}\right)\cos\varphi\right. \\ &\qquad \left. + \nu\left[\frac{1}{r_1}\left(\frac{\dot{w} - v}{r_1}\right)^{\cdot}\right]\right\} \end{aligned} \tag{18e}$$

$$\begin{aligned} M_{\varphi\theta} = K(1-\nu)\kappa_{\varphi\theta} &= K(1-\nu)\left[\frac{1}{2}\left(-\frac{1}{r_1}\dot{\omega}_\varphi - \frac{1}{r}\omega_\theta' + \frac{1}{r}\omega_\varphi\cos\varphi\right)\right] \\ &= K(1-\nu)\left[\frac{1}{2r_1}\left(\frac{w' - u\sin\varphi}{r}\right)^{\cdot} + \frac{1}{2r}\left(\frac{\dot{w} - v}{r_1}\right)'\right. \\ &\qquad \left. - \frac{1}{2r}\left(\frac{w' - u\sin\varphi}{r}\right)\cos\varphi\right] \end{aligned} \tag{18f}$$

where the extensional and bending stiffnesses are given by

$$D = \frac{Eh}{1 - \nu^2} \quad , \tag{19}$$

and

$$K = \frac{Eh^3}{12(1 - \nu^2)} \quad , \tag{20}$$

respectively.

(Note that in a Donnell-type analysis, u and v are discarded from the formulas for κ_φ , κ_θ , and $\kappa_{\varphi\theta}$) .

The elastic law given by Eqs. (18) is the same as that derived by Reissner (Ref. 12) and Gravina (Ref. 20) who used the methods of differential geometry.

The elastic law gives six equations for nine unknowns (N_φ , N_θ , $N_{\varphi\theta}$, M_φ , M_θ , $M_{\varphi\theta}$, u , v , w) and therefore are not sufficient to determine the unknowns. The additional equations needed are, of course, found from the conditions of equilibrium which will be derived in the next chapter.

IV

EQUATIONS OF EQUILIBRIUM

In deriving the equations of equilibrium for the linear bending analysis of shells it is irrelevant as to whether the forces and moments are assumed to act on a deformed or on an undeformed element of the shell. In shell stability problems, however, the equations of equilibrium must be written for an element of the deformed (buckled) shell. These equations will be derived for a general shell of revolution in this chapter.

1. Equilibrium Equations for the Stability of a Shell of Revolution

In this section, equations of equilibrium are derived based on the classical assumption that the effects of prebuckling rotations may be neglected.

Figure 1 shows the middle surface of a differential element of the shell in the prebuckled state. At the center of the element, point P , there is shown an orthogonal right-handed system of axes X , Y , and Z with the X axis in the direction of the tangent to the meridian at P , the Y axis in the direction of the tangent to the parallel circle at P , and the Z axis in the direction of the outward normal at P . The force and moment equations of equilibrium for a differential element of the buckled shell will be written with respect to the X , Y , and Z directions. The terms that contribute to the equilibrium equations may be divided into four groups denoted as Group 0, Group 1, etc. Group 0 contains only the prebuckling quantities ($N_{\varphi 0}$, $N_{\theta 0}$, $N_{\varphi \theta 0}$, $N_{\theta \varphi 0}$, p_{φ} , p_{θ} , p_z) . Due to prebuckling equilibrium, the net contribution of these terms to the equilibrium equations for the buckled shell must be zero. Group 1 contains only the incremental quantities which develop as the shell passes from the prebuckled state to the buckled state. In a linear stability analysis, only terms which are linear in the (infinitesimal) incremental quantities are retained. These terms are the

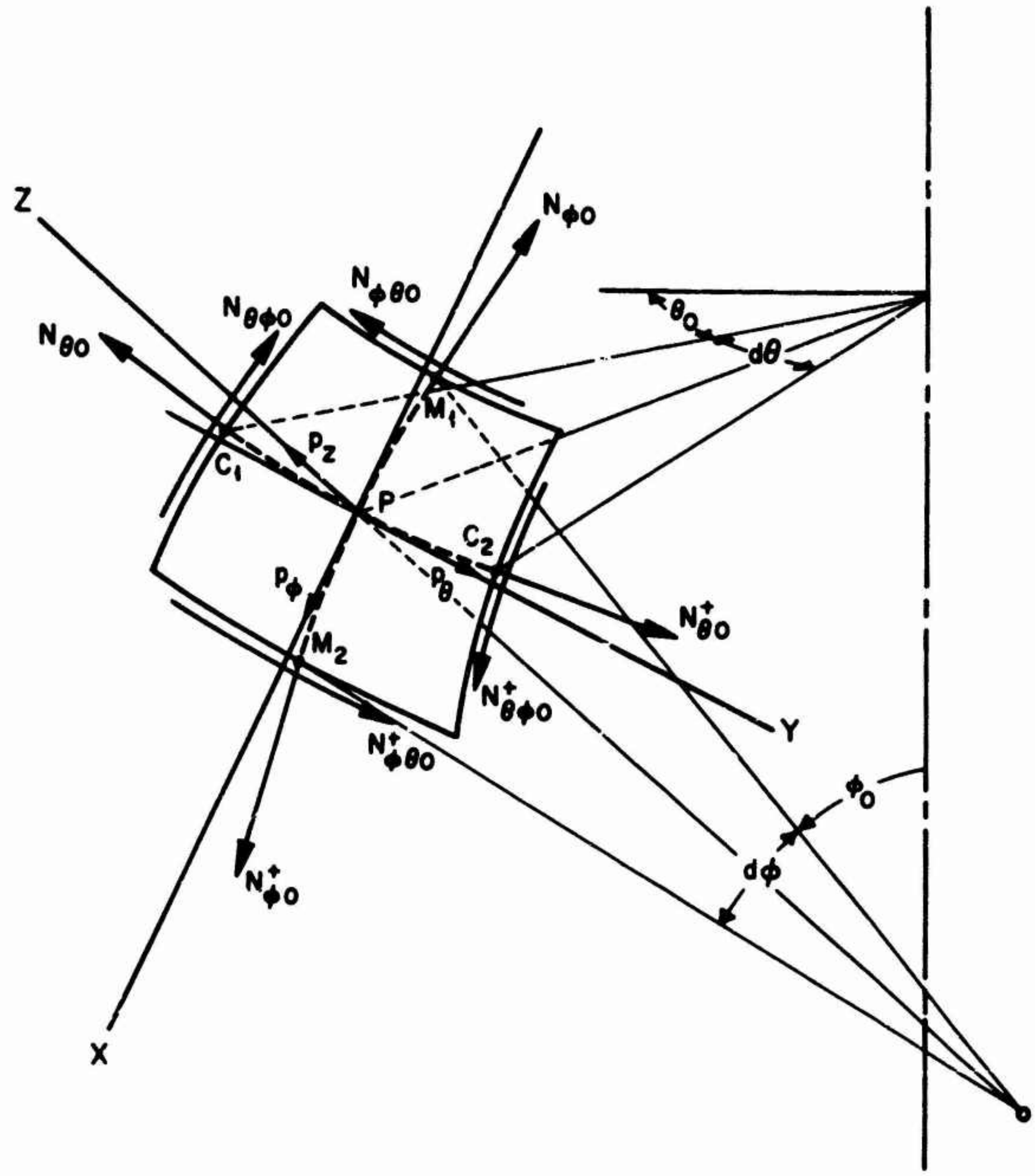


Fig. 1 Element of Prebuckled Shell

same as those present in the linear bending analysis of shells. The Group $\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$ terms arise because, in the buckled state, the prebuckling quantities

$$\begin{Bmatrix} N_{\phi 0}\ ,\ N_{\theta 0}\ ,\ N_{\phi\theta 0}\ ,\ N_{\theta\phi 0} \\ p_{\phi}\ ,\ p_{\theta}\ ,\ p_{z} \end{Bmatrix}$$

act on a deformed element. These terms consist of products of a (finite) prebuckling quantity and an (infinitesimal) incremental displacement or its derivative. The contributions to the equilibrium equations of terms which are linear in the incremental quantities, i.e., the terms in Group 1, 2, and 3, are given in the next three subsections.

1.1 Group 1: Incremental Stress Resultants

In the determination of the contributions of the incremental stress resultants N_{ϕ} , . . . , Q_{θ} , . . . , $M_{\theta\phi}$ to the equations of equilibrium for the buckled shell, it is irrelevant as to whether the stress resultants are assumed to act on a buckled or on an unbuckled element. This is because the equilibrium equations for the buckled element differ from those for the unbuckled element only in terms which are nonlinear in the incremental quantities, and, since the incremental quantities are infinitesimals, these nonlinear terms vanish. Thus, the contributions of the incremental stress resultants may be obtained from a consideration of equilibrium for an undeformed element. The results of such a consideration follow (Ref. 13):

$$\Sigma F_1' = (rN_{\varphi})^{\cdot} + r_1 N_{\theta\varphi}' - r_1 N_{\theta} \cos\varphi - rQ_{\varphi} \tag{1a}$$

$$\Sigma F_2' = (rN_{\varphi\theta})^{\cdot} + r_1 N_{\theta}' + r_1 N_{\varphi\theta} \cos\varphi - r_1 Q_{\theta} \sin\varphi \tag{1b}$$

$$\Sigma F_3' = - r_1 N_{\theta} \sin\varphi - rN_{\varphi} - r_1 Q_{\theta}' - (rQ_{\varphi})^{\cdot} \tag{1c}$$

$$\Sigma M_1' = (rM_{\varphi\theta})^{\cdot} + r_1 M_{\theta}' + r_1 M_{\theta\varphi} \cos\varphi - rr_1 Q_{\theta} \tag{1d}$$

$$\Sigma M_2' = - (rM_{\varphi})^{\cdot} - r_1 M_{\theta\varphi}' + r_1 M_{\theta} \cos\phi + rr_1 Q_{\phi} \ . \tag{1e}$$

Note that a common factor $d\theta\, d\varphi$ has been omitted in Eqs. (1). This factor will also be omitted in the sequel when other contributions to the equilibrium equations are obtained.

1.2 Group 2: Prebuckling Stress Resultants

When the shell buckles, the prebuckling stress resultants rotate and thereby develop components which contribute to the equilibrium equations. These components, which consist of products of a prebuckling stress resultant and an incremental rotation, will now be determined for each of the prebuckling stress resultants $N_{\varphi 0}$, $N_{\theta 0}$, $N_{\theta\varphi 0}$, and $N_{\varphi\theta 0}$.

1.2.1 $N_{\varphi 0}$

When the shell passes from the prebuckled state to the buckled state, the meridional force $N_{\varphi 0} r d\theta$, acting at point M_1, of the section $\varphi = \varphi_0$ (Fig. 1 and Fig. 2a), participates in the incremental rotation ω_θ and therefore develops a component

$$K_1 = rN_{\varphi 0}\omega_\theta d\theta$$

which points in the direction of the outward normal (to the prebuckled shell) at M_1 . Similarly, the meridional force $N_{\varphi 0} r d\theta + (N_{\varphi 0} r d\theta)^{\cdot} d\varphi$, acting at point M_2 of the section $\varphi = \varphi_0 + d\varphi$, develops a normal (or radial) component

$$K_1^{+} = \left[N_{\varphi 0} r d\theta + (N_{\varphi 0} r d\theta)^{\cdot} d\varphi\right]\left[\omega_\theta + \omega_\theta^{\cdot} d\varphi\right] = rN_{\varphi 0}\omega_\theta d\theta + (rN_{\varphi 0}\omega_\theta d\theta)^{\cdot} d\varphi$$

which points toward the center of curvature of the meridian at point M_2 . The contributions of the components K_1 and K_1^{+} to the equilibrium equations can easily be seen from Fig. 2b. Thus, due to the difference in their directions, K_1 and K_1^{+} contribute an amount $-rN_{\varphi 0}\omega_\theta$ to the equilibrium of forces in the

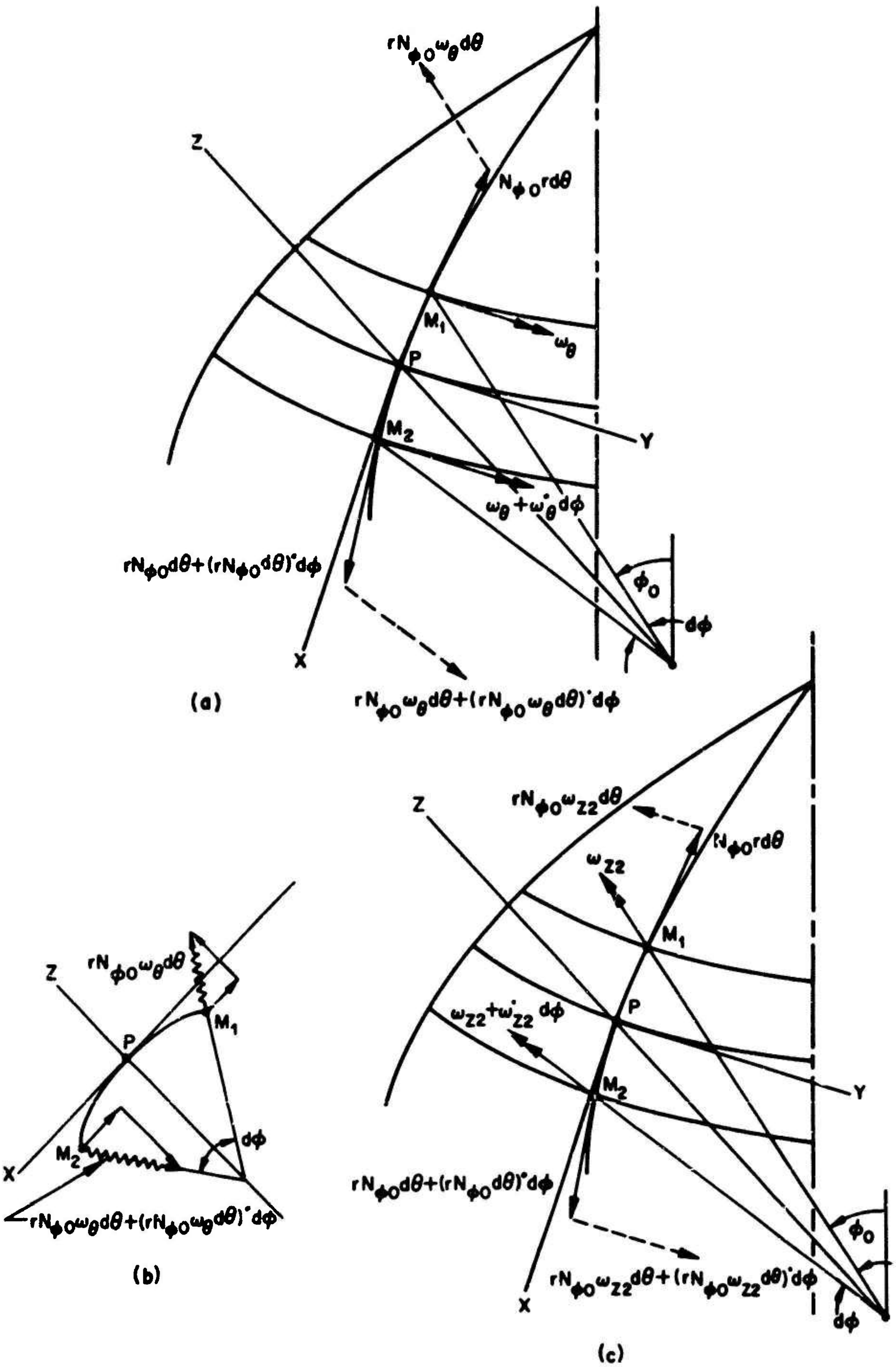


Fig. 2 Contribution of $N_{\phi 0}$

direction of the tangent to the meridian at P , i.e., in the X direction; and due to a difference in the magnitudes of K_1 and K_1^+ , there is a contribution $-(rN_{\varphi 0}\omega_\theta)^{\cdot}$ to the equilibrium of forces in the direction of the normal at P , i.e., in the Z direction.

Due to the incremental rotation ω_{z2} , the meridional force $N_{\varphi 0} r d\theta$, at the section $\varphi = \varphi_0$, develops a component

$$K_2 = rN_{\varphi 0}\omega_{z2} d\theta$$

which points in the negative Y direction (see Fig. 2c), and the meridional force $rN_{\varphi 0}d\theta + (rN_{\varphi 0}d\theta)^{\cdot} d\varphi$, at the section $\varphi = \varphi_0 + d\varphi$, develops a component

$$K_2^+ = \left[rN_{\varphi 0}d\theta + (rN_{\varphi 0}d\theta)^{\cdot} d\varphi\right]\left[\omega_{z2} + \dot{\omega}_{z2}d\varphi\right] = rN_{\varphi 0}\omega_{z2}d\theta + (rN_{\varphi 0}\omega_{z2}d\theta)^{\cdot} d\varphi$$

which points in the positive Y direction. Since both K_2 and K_2^+ act in the Y direction, there is a contribution $+(rN_{\varphi 0}\omega_{z2})^{\cdot}$ to equilibrium of forces in the Y direction as a result in the difference in magnitudes of K_2 and K_2^+ .

The stress resultant $N_{\varphi 0}$ does not develop any components due to the rotation ω_φ since the direction of $N_{\varphi 0}$ is parallel to that of ω_φ .

Thus, the contributions of the prebuckling stress resultant $N_{\varphi 0}$ to the equations of equilibrium of forces in the X , Y , and Z directions are $-rN_{\varphi 0}\omega_\theta$, $+(rN_{\varphi 0}\omega_{z2})^{\cdot}$, and $-(rN_{\varphi 0}\omega_\theta)^{\cdot}$, respectively.

1.2.2 $\underline{N_{\theta 0}}$

The hoop force $N_{\theta 0} r_1 d\varphi$, acting at point C_1 of the section $\theta = \theta_C$ (Fig. 3a), participates in the incremental rotation ω_φ and develops a component

$$K_3 = r_1 N_{\theta 0}\omega_\varphi d\varphi$$

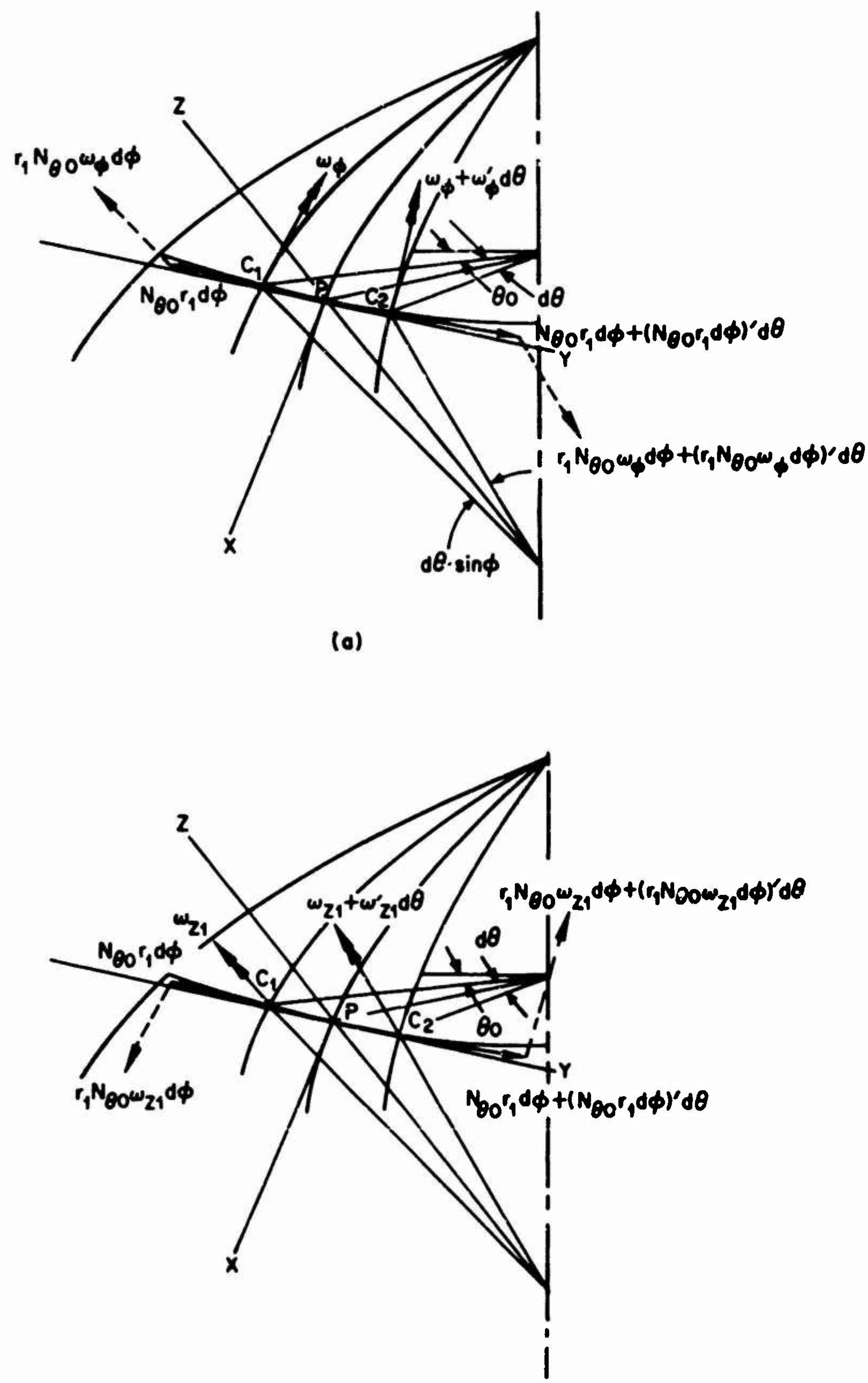


Fig. 3 Contribution of $N_{\theta 0}$

which points in the direction of the outward normal at C_1 . Similarly, the hoop force $N_{\theta 0} r_1 d\varphi + (N_{\theta 0} r_1 d\varphi)' d\theta$, acting at point C_2 of the section $\theta = \theta_0 + d\theta$, develops a normal component

$$K_3^+ = \left| r_1 N_{\theta 0} d\varphi + (r_1 N_{\theta 0} d\varphi)' d\theta \right| \left| \omega_\varphi + \omega_\varphi' d\theta \right|$$

$$= r_1 N_{\theta 0} \omega_\varphi d\varphi + (r_1 N_{\theta 0} \omega_\varphi d\varphi)' d\theta$$

which points toward the center of curvature of the meridian at point C_2 . From Fig. 3a we see that due to a difference in their directions, the forces K_3 and K_3^+ contribute an amount

$$-(r_1 N_{\theta 0} \omega_\varphi \sin \varphi)$$

to the equilibrium of forces in the Y direction, and that due to a difference in the magnitudes of K_3 and K_3^+ there is a contribution

$$-r_1 (N_{\theta 0} \omega_\varphi)'$$

to the equilibrium of forces in the Z direction.

Due to the incremental rotation ω_{z1} , the hoop force $N_{\theta 0} r_1 d\varphi$, at $\theta = \theta_0$, develops a component

$$K_4 = r_1 N_{\theta 0} \omega_{z1} d\varphi$$

which acts along the tangent to the meridian at $\theta = \theta_0$ and which points in the direction of increasing φ (Fig. 3b), and the hoop force $N_{\theta 0} r_1 d\varphi + (N_{\theta 0} r_1 d\varphi)' d\theta$, at the section $\theta = \theta_0 + d\theta$, develops a component

$$K_4^+ = \left| r_1 N_{\theta 0} d\varphi + (r_1 N_{\theta 0} d\varphi)' d\theta \right| \left| \omega_{z1} + \omega_{z1}' d\theta \right|$$

$$= r_1 N_{\theta 0} \omega_{z1} d\varphi + (r_1 N_{\theta 0} \omega_{z1} d\varphi)' d\theta$$

which acts along the tangent to the meridian at $\theta = \theta_0 + d\theta$ and which points in the direction of decreasing φ . From Fig. 3b we see that the components K_4 and K_4^+ contribute an amount

$$-r_1(N_{\theta 0}\omega_{z1})'$$

to the equilibrium of forces in the X direction and an amount

$$-r_1 N_{\theta 0}\omega_{z1}\cos\phi$$

to the equilibrium of forces in the Y direction.

The stress resultant $N_{\theta 0}$ does not develop any components due to the rotation ω_θ .

Thus, the contributions of the prebuckling stress resultant $N_{\theta 0}$ to the equations of equilibrium of forces with respect to the X , Y , and Z directions are $-r_1(N_{\theta 0}\omega_{z1})'$, $-r_1 N_{\theta 0}\omega_\varphi \sin\varphi - r_1 N_{\theta 0}\omega_{z1}\cos\varphi$, and $-r_1(N_{\theta 0}\omega_\varphi)'$, respectively.

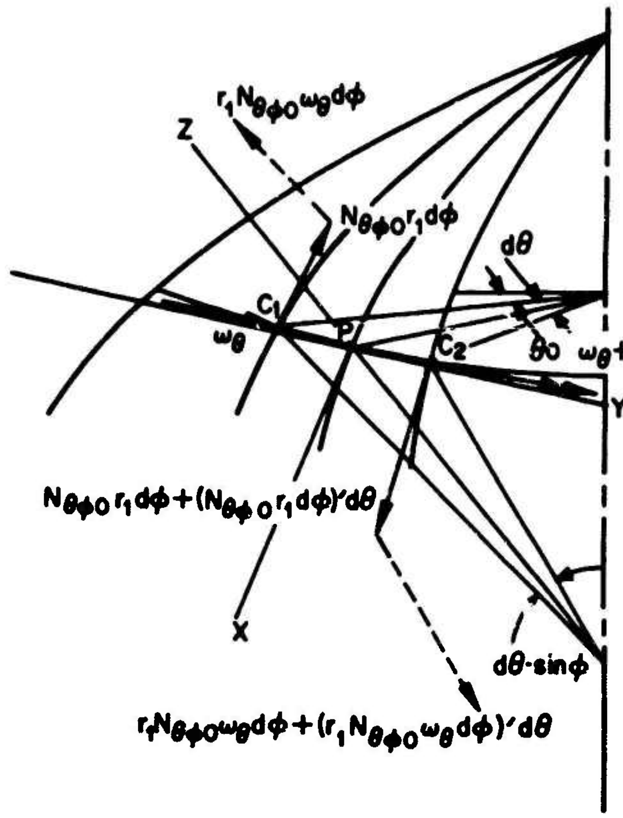
1.2.3 $\underline{N_{\theta\varphi 0} \text{ and } N_{\varphi\theta 0}}$

The contributions of the prebuckling stress resultants $N_{\theta\varphi 0}$ and $N_{\varphi\theta 0}$ to the equations of equilibrium for an element of the deformed shell can be obtained in the same way as the contributions of $N_{\varphi 0}$ and $N_{\theta 0}$. Thus, according to Figs. 4 and 5, we obtain for equilibrium of forces in the X , Y , and Z directions the contributions

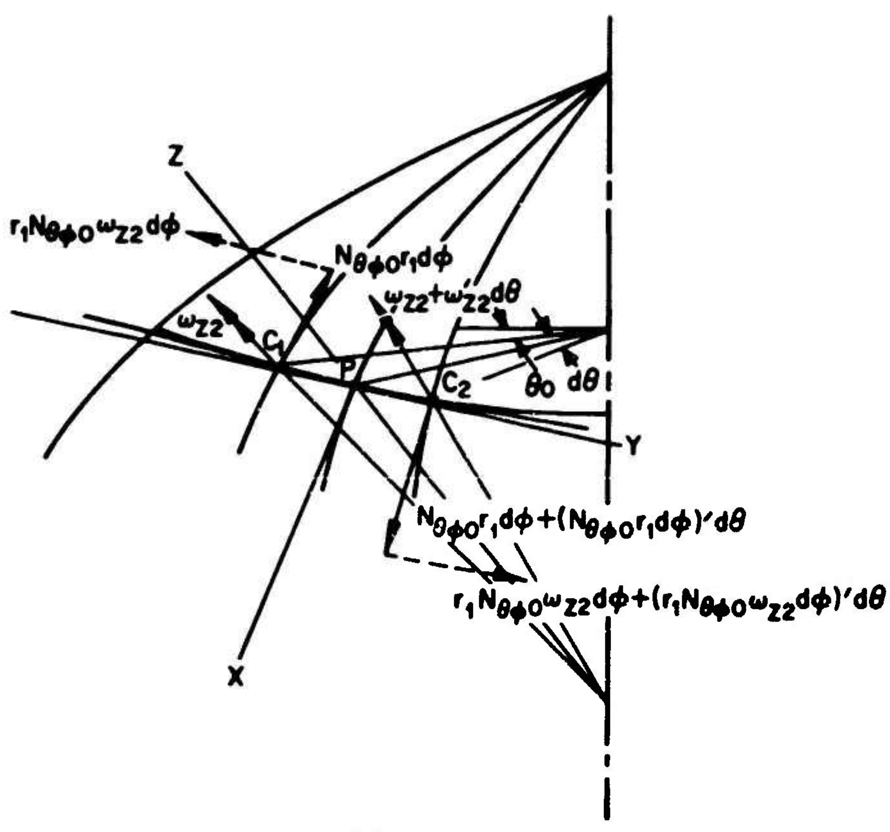
$-r_1 N_{\theta\varphi 0}\omega_{z2}\cos\varphi - rN_{\varphi\theta 0}\omega_\theta - (rN_{\varphi\theta 0}\omega_{z1})^{\cdot}$,

$-r_1 N_{\theta\varphi 0}\omega_\theta \sin\varphi + r_1(N_{\theta\varphi 0}\omega_{z2})'$, and

$-r_1(N_{\theta\varphi 0}\omega_\theta)' - r_1 N_{\theta\varphi 0}\omega_{z2}\sin\varphi - (rN_{\varphi\theta 0}\omega_\varphi)^{\cdot} + rN_{\varphi\theta 0}\omega_{z1}$, respectively.

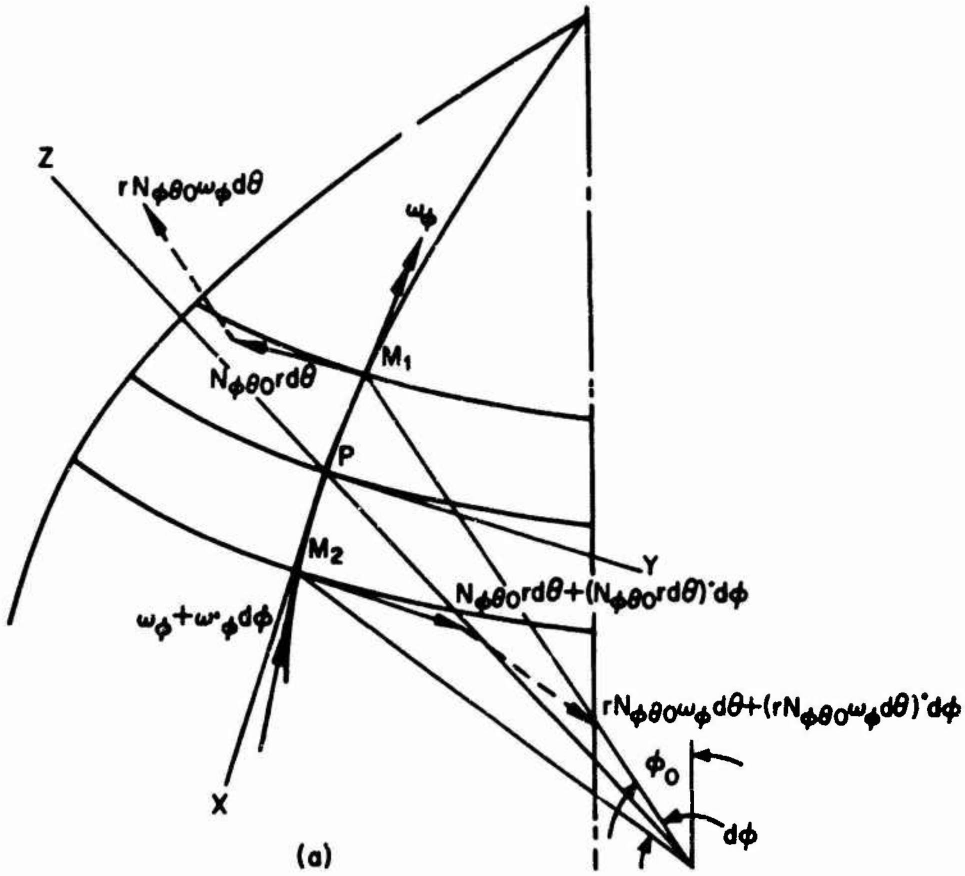


(a)



(b)

Fig. 4 Contribution of $N_{\theta\phi 0}$



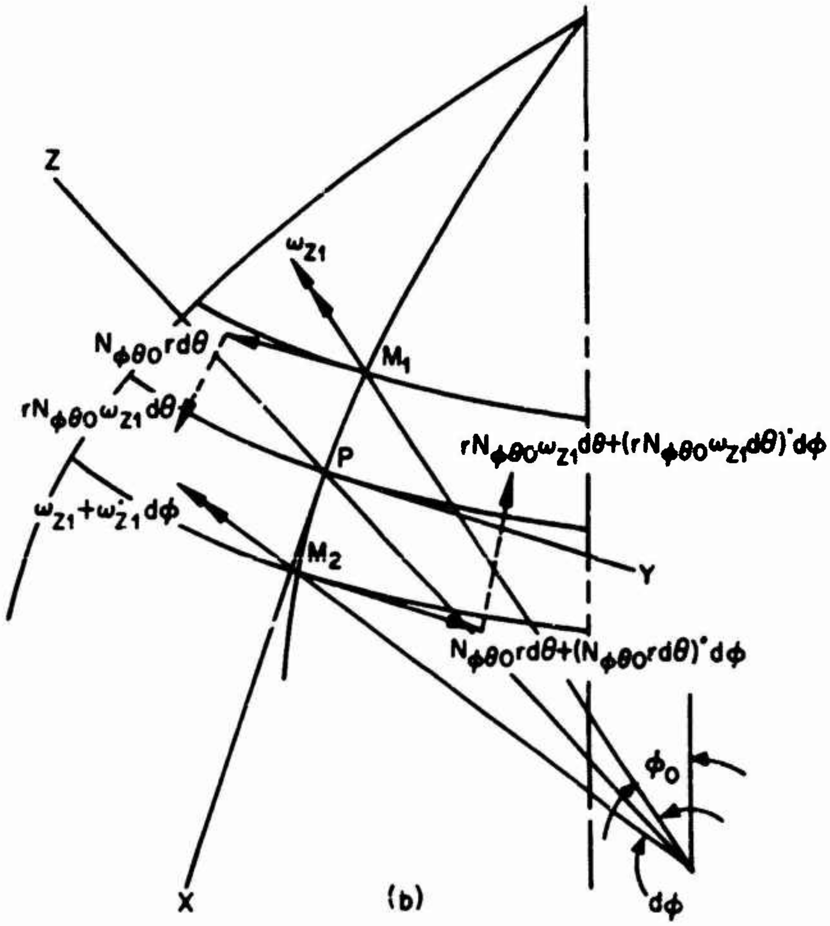


Fig. 5 Contribution of $N_{\phi\theta 0}$

1.2.4 Summary of Group 2 Contributions

The contributions of the prebuckling stress resultants $N_{\varphi 0}$, $N_{\theta 0}$, $N_{\theta\varphi 0}$, and $N_{\varphi\theta 0}$ to the equations of equilibrium for an element of the deformed shell are

$$\Sigma F_1^2 = -rN_{\varphi 0}\omega_\theta - r_1(N_{\theta 0}\omega_{z1})' - r_1 N_{\theta\varphi 0}\omega_{z2}\cos\varphi - rN_{\varphi\theta 0}\omega_\varphi - (rN_{\varphi\theta 0}\omega_{z1})^{\cdot} \qquad (2a)$$

$$\Sigma F_2^2 = +(rN_{\varphi 0}\omega_{z2})^{\cdot} - r_1 N_{\theta 0}\omega_\varphi \sin\varphi - r_1 N_{\theta 0}\omega_{z1}\cos\varphi - r_1 N_{\theta\varphi 0}\omega_\theta \sin\varphi + r_1(N_{\theta\varphi 0}\omega_{z2})' \qquad (2b)$$

$$\Sigma F_3^2 = -(rN_{\varphi 0}\omega_\theta)^{\cdot} - r_1(N_{\theta 0}\omega_\varphi)' - r_1(N_{\theta\varphi 0}\omega_\theta)' - r_1 N_{\theta\varphi 0}\omega_{z2}\sin\varphi - (rN_{\varphi\theta 0}\omega_\varphi)^{\cdot} + rN_{\varphi\theta 0}\omega_{z1} \cdot \qquad (2c)$$

1.3 Group 3: Applied Loads

The contributions of the applied loads to the equilibrium equations will now be determined. These contributions arise because, in the buckled shell, the applied loads act on an element which has been deformed by the incremental displacements.

Let the components of the applied load per unit area of the middle surface of the prebuckled shell be denoted by p_φ , p_θ , and p_z as shown in Fig. 1. The loads p_φ and p_θ are taken positive in the direction of increasing φ and θ , respectively; p_z is positive when it points away from the center of curvature of the meridian. Now the statical approach used here to obtain the buckling loads is applicable only to conservative systems (Ref. 21). Hence, the load components are assumed to be conservative; for example, constant directional loads (dead weight loads) or hydrostatic pressure loads.

Since all stability problems involve considerations of the deformed structure, it is necessary to specify precisely the character of the applied loads; i.e., the way the applied loads behave as the shell deforms. Thus, for example, the stability equations for a constant directional pressure loading will differ from those for a hydrostatic pressure loading. It is conceivable that small changes in the nature of the applied loads might have an appreciable effect on the magnitude of the buckling load; indeed, for buckling of a ring the difference amounts to 33% (Ref. 22). Two types of pressure loadings will be considered here: constant directional pressure loading (dead weight loading) and hydrostatic pressure loading. A constant directional pressure loading is such that the total force acting on a shell element does not vary in magnitude or in direction as the element deforms. Thus, a constant directional force does not contribute (explicitly) to the equilibrium equations. A hydrostatic pressure loading is such that the total force acting on an element of the shell is always proportional to the actual size of the element and is always directed normal to the element. Then, due to the incremental rotations ω_θ and ω_φ, the hydrostatic pressure force $rr_1p_z d\theta d\varphi$ acting on an element of the shell develops the components $rr_1p_z\omega_\theta d\theta d\varphi$ and $rr_1p_z\omega_\varphi d\theta d\varphi$ which point in the positive X and Y directions, respectively. In addition, due to the stretching of the middle surface during buckling, there is a component, $+rr_1p_z(\overline{\epsilon}_\theta + \overline{\epsilon}_\varphi)\, d\theta\, d\varphi$ which points in the positive Z direction.

Thus, the explicit contributions of the applied loads to the equations of equilibrium for an element of the deformed shell may be written as

$$\Sigma F_1^3 = +\delta_{ph}(rr_1p_z\omega_\theta) \tag{3a}$$

$$\Sigma F_2^3 = +\delta_{ph}(rr_1p_z\omega_\varphi) \tag{3b}$$

$$\Sigma F_3^3 = +\delta_{ph}\, rr_1p_z(\overline{\epsilon}_\theta + \overline{\epsilon}_\varphi) \tag{3c}$$

where

$$\delta_{ph} = \begin{cases} 1, \text{ for hydrostatic pressure loading} \\ 0, \text{ for constant directional pressure} \end{cases} \tag{4}$$

1.4 Summary of Results

By adding the contributions given by Eqs. (1) through (3), we arrive at the following equations of equlibrium for an element of the buckled shell:

$$\begin{aligned} \Sigma F_1 = {} & (rN_\varphi)^{\cdot} + r_1 N'_{\theta\varphi} - r_1 N_\theta \cos\varphi + \left| -rQ_\varphi - rN_{\varphi 0}\omega_\theta - rN_{\varphi\theta 0}\omega_\varphi \right| \\ & + \left[-r_1(N_{\theta 0}\omega_{z1})' - (rN_{\varphi\theta 0}\omega_{z1})^{\cdot} - r_1 N_{\theta\varphi 0}\omega_{z2}\cos\varphi \right] \\ & + \delta_{ph}\, r r_1 p_z \omega_\theta = 0 \end{aligned} \tag{5a}$$

$$\begin{aligned} \Sigma F_2 = {} & (rN_{\varphi\theta})^{\cdot} + r_1 N'_\theta + r_1 N_{\theta\varphi}\cos\varphi + \left| -r_1 Q_\theta \sin\varphi - r_1 N_{\theta\varphi 0}\omega_\theta \sin\varphi \right. \\ & \left. - r_1 N_{\theta 0}\omega_\varphi \sin\varphi \right| + \left| -r_1 N_{\theta 0}\omega_{z1}\cos\varphi + (rN_{\varphi 0}\omega_{z2})^{\cdot} \right. \\ & \left. + r_1(N_{\theta\varphi 0}\omega_{z2})' \right| + \delta_{ph}\, r r_1 p_z \omega_\varphi = 0 \end{aligned} \tag{5b}$$

$$\begin{aligned} \Sigma F_3 = {} & -r_1 N_\theta \sin\varphi - rN_\varphi - r_1 Q'_\theta - (rQ_\varphi)^{\cdot} - (rN_{\varphi 0}\omega_\theta)^{\cdot} - r_1(N_{\theta\varphi 0}\omega_\theta)' \\ & - r_1(N_{\theta 0}\omega_\varphi)' - (rN_{\varphi\theta 0}\omega_\varphi)^{\cdot} + \left| rN_{\varphi\theta 0}\omega_{z1} - r_1 N_{\theta\varphi 0}\omega_{z2}\sin\varphi \right| \\ & + \delta_{ph}\, r r_1 p_z(\bar{\epsilon}_\varphi + \bar{\epsilon}_\theta) = 0 \end{aligned} \tag{5c}$$

$$\Sigma M_1 = (rM_{\varphi\theta})^{\cdot} + r_1 M'_\theta + r_1 M_{\theta\varphi}\cos\varphi - r r_1 Q_\theta = 0 \tag{5d}$$

$$\Sigma M_2 = -(rM_\varphi)^{\cdot} - r_1 M'_{\theta\varphi} + r_1 M_\theta \cos\phi + r r_1 Q_\phi = 0 \tag{5e}$$

If effects of rotations around the normal are negligible, then the terms in the straight brackets in Eqs. (5) are omitted. In a Donnell type analysis, the terms in the braces and brackets in Eqs. (5) are neglected.

2. Nonlinear Equations of Equilibrium for a Shell of Revolution

The stability equations presented in the preceding section were based on the classical assumption that the prebuckling rotations could be neglected. In this section, we present stability equations which include the effects of prebuckling rotations. These equations are obtained through specialization of the nonlinear equations of equilibrium which will be derived here. To obtain the stability equations, we replace each unknown quantity $(\bar{\ })$ in the nonlinear equations of equilibrium by $(\ )_0 + (\ )$, where $(\ )_0$ represents the prebuckling value of the quantity and $(\ )$ represents the incremental value of the quantity which develops during buckling. Then in the resulting equations, the terms containing only prebuckling quantities may be subtracted out by virtue of prebuckling equilibrium, and nonlinear terms in the (infinitesimal) incremental quantities may be disregarded. The result is, of course, the linear stability equations. Actually, only the nonlinear force equilibrium equations will be presented here. The nonlinear moment equilibrium equations can be obtained from considerations identical to those used in arriving at the nonlinear force equilibrium equations. However, for the case of a thin shell, the nonlinear terms in the moment equilibrium equations are negligible.

The stress resultants, rotations, and strains which develop as the shell deforms due to the applied loads p_ϕ, p_θ, and p_z are denoted by $\bar{N}_\phi$, $\bar{N}_\theta$, ..., $\bar{Q}_\phi$, ..., $\bar{M}_{\theta\phi}$, $\bar{\omega}_\phi$, $\bar{\omega}_\theta$, $\bar{\omega}_{z1}$, $\bar{\omega}_{z2}$, $\bar{\bar{\epsilon}}_\phi$, and $\bar{\bar{\epsilon}}_\theta$. These quantities are zero when the shell is unloaded. Linearized relations between these quantities and the displacement components of the middle surface can be obtained from Eqs. (III-8), (III-9), (III-11), and (III-18) providing the displacement components appearing in these equations are measured from the unloaded shell.

Figure 6 shows the middle surface of a differential element of the unloaded shell. At the center of the element, point P* , there is shown an orthogonal

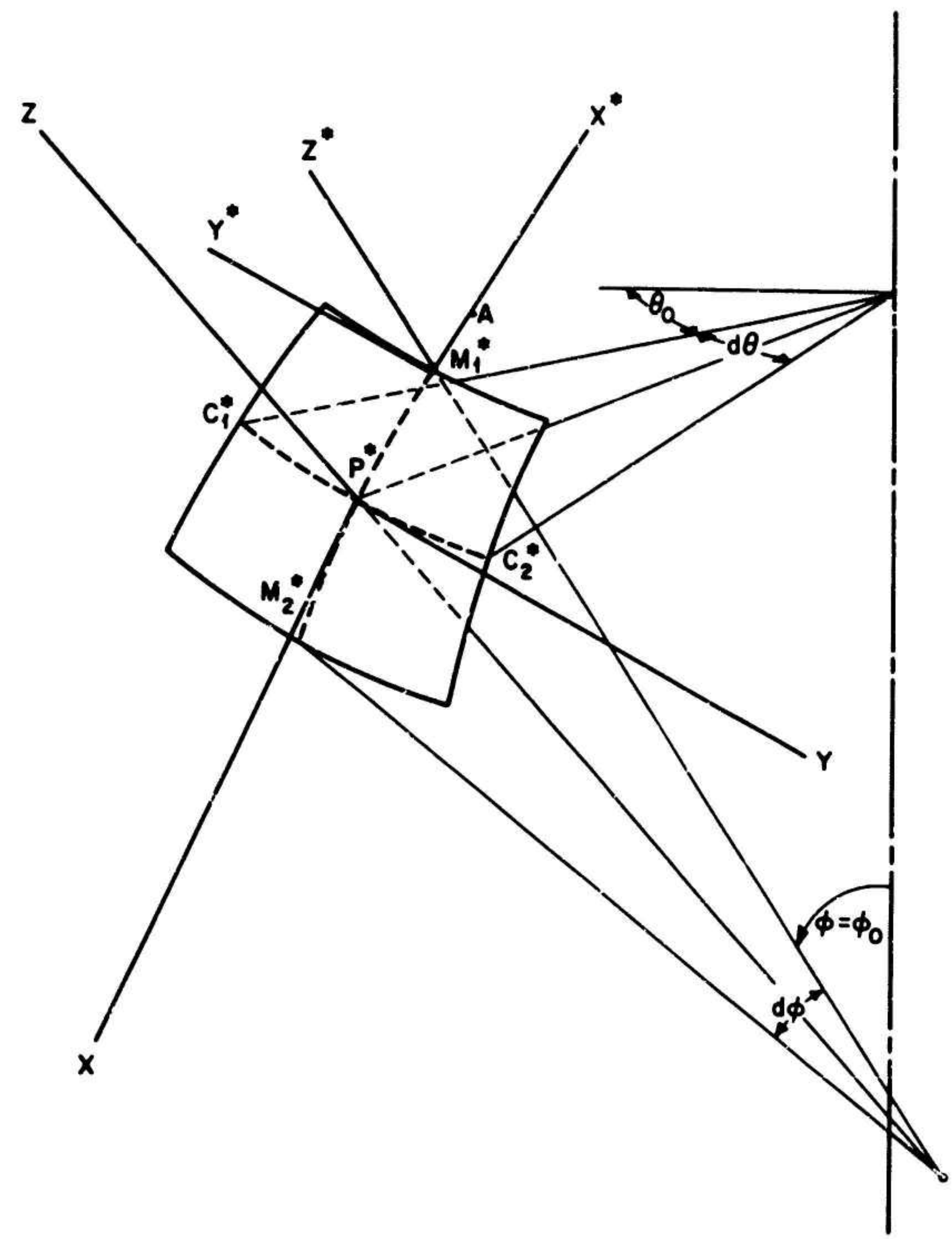


Fig. 6 Shell Eiement

right-handed system of axes X , Y , and Z with the X axis in the direction of the tangent to the meridian at P* , the Y axis in the direction of the tangent to the parallel circle at P* , and the Z axis in the direction of the outward normal at P* . The equations of equilibrium for a differential element of the deformed shell will be written with respect to the X , Y , and Z axes. Let M_1^* denote the point of intersection of the meridian through P* and the upper parallel circle (see Fig. 6). At the point M_1^* let the directions of the tangent to the meridian, the tangent to the parallel circle, and the normal be given by the X* , Y* , and Z* axes, respectively. Now due to the applied loads, the line M_1^*A , which is tangent to the meridian of the unloaded shell at M_1^* , acquires the new direction X** as shown in Fig. 7a. The meridional force $r\bar{N}_\varphi d\theta$, at point M_1^* of the section $\varphi = \varphi_0$, is defined to act in the X** direction. Then according to Fig. 7a, the components of $\bar{N}_\varphi$ along the X* , Y* , and Z* directions are

$$r\bar{N}_\varphi \cos \bar{\omega}_\theta \cos \bar{\omega}_{z2}\, d\theta \quad ,$$

$$r\bar{N}_\varphi \cos \bar{\omega}_\theta \sin \bar{\omega}_{z2}\, d\theta \quad , \text{ and}$$

$$r\bar{N}_\varphi \sin \bar{\omega}_\theta\, d\theta \quad , \text{ respectively. These components}$$

are shown in Fig. 7b. The components of the meridional force $r\bar{N}_\varphi d\theta + (r\bar{N}_\varphi d\theta)^{\cdot} d\varphi$, which acts at point M_2^* of the section $\varphi = \varphi_0 + d\varphi$, are also shown in Fig. 7b. Next, with $\sin \bar{\omega}_i \rightarrow \bar{\omega}_i$ and $\cos \bar{\omega}_i \rightarrow 1$, and from Fig. 7b, the contributions of the stress resultant $\bar{N}_\varphi$ to the equilibrium equations written with respect to the X , Y , and Z directions are $(r\bar{N}_\varphi)^{\cdot} - r\bar{N}_\varphi \bar{\omega}_\theta$, $(r\bar{N}_\varphi \bar{\omega}_{z2})^{\cdot}$, and $-r\bar{N}_\varphi - (r\bar{N}_\varphi \bar{\omega}_\theta)^{\cdot}$, respectively.

The contributions of the other stress resultants may be obtained in the same way as the contributions of $\bar{N}_\phi$ were obtained. Thus, according to

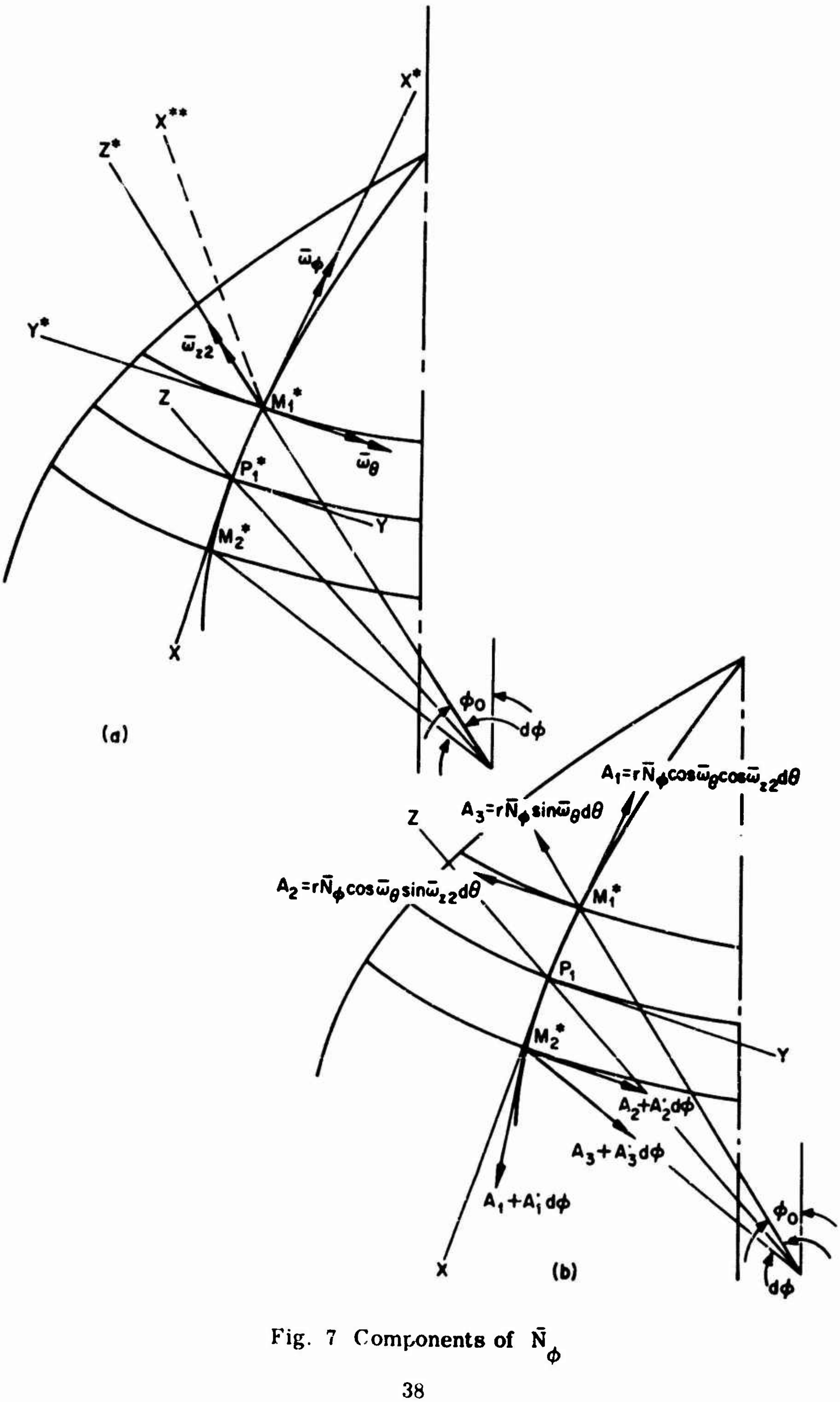


Fig. 7 Components of $\bar{N}_\phi$

Figs. 8 to 10, the nonlinear equations of equilibrium for a differential element of the deformed shell are

$$\Sigma F_1 = (r\overline{N}_\phi)^{\cdot} + (r_1\overline{N}_{\theta\phi})' - r_1\overline{N}_\theta \cos\phi - r\overline{Q}_\phi - r\overline{N}_\phi\overline{\omega}_\theta - r\overline{N}_{\phi\theta}\overline{\omega}_\phi$$
$$- (r\overline{N}_{\phi\theta}\overline{\omega}_{z1})^{\cdot} - (r_1\overline{N}_\theta\overline{\omega}_{z1})' - r_1\overline{N}_{\theta\phi}\overline{\omega}_{z2}\cos\phi$$
$$+ rr_1p_\phi + \delta_{ph}(rr_1p_z\overline{\omega}_\theta) = 0 \tag{6a}$$

$$\Sigma F_2 = (r_1\overline{N}_\theta)' + (r\overline{N}_{\phi\theta})^{\cdot} + r_1\overline{N}_{\theta\phi}\cos\phi - r_1\overline{Q}_\theta \sin\phi - r_1\overline{N}_{\theta\phi}\overline{\omega}_\theta \sin\phi$$
$$- r_1\overline{N}_\theta\overline{\omega}_\phi \sin\phi + (r_1\overline{N}_{\theta\phi}\overline{\omega}_{z2})' + (r\overline{N}_\phi\overline{\omega}_{z2})^{\cdot}$$
$$- r_1\overline{N}_\theta\overline{\omega}_{z1}\cos\phi + rr_1p_\theta + \delta_{ph}(rr_1p_z\overline{\omega}_\phi) = 0 \tag{6b}$$

$$\Sigma F_3 = -r\overline{N}_\phi - r_1\overline{N}_\theta \sin\phi - (r\overline{Q}_\phi)^{\cdot} - (r_1\overline{Q}_\theta)' - (r\overline{N}_\phi\overline{\omega}_\theta)^{\cdot} - (r_1\overline{N}_{\theta\phi}\overline{\omega}_\theta)'$$
$$- (r\overline{N}_{\phi\theta}\overline{\omega}_\phi)^{\cdot} - (r_1\overline{N}_\theta\overline{\omega}_\phi)' + r\overline{N}_{\phi\theta}\overline{\omega}_{z1} - r_1\overline{N}_{\theta\phi}\overline{\omega}_{z2}\sin\phi$$
$$+ rr_1p_z + \delta_{ph}rr_1p_z(\bar{\epsilon}_\theta + \bar{\epsilon}_\phi) = 0 \tag{6c}$$

$$\Sigma M_1 = (r\overline{M}_{\phi\theta})^{\cdot} + (r_1\overline{M}_\theta)' + r_1\overline{M}_{\theta\phi}\cos\phi - rr_1\overline{Q}_\theta = 0 \tag{6d}$$

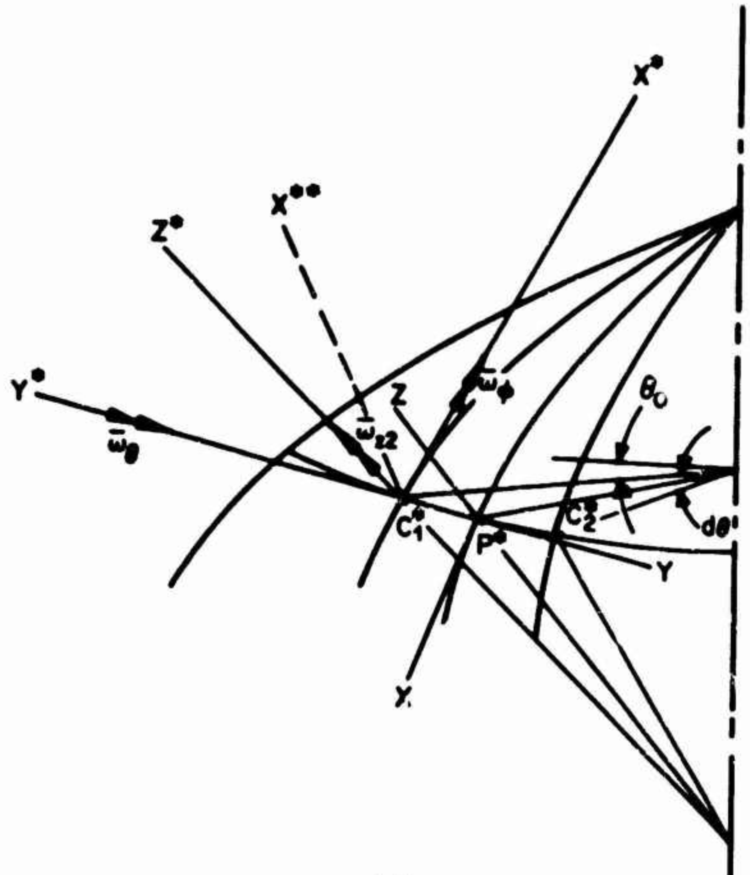
$$\Sigma M_2 = -(r\overline{M}_\phi)^{\cdot} - (r_1\overline{M}_{\theta\phi})' + r_1\overline{M}_\theta \cos\phi + rr_1\overline{Q}_\phi = 0 \quad . \tag{6e}$$

The equations governing the stability of a shell of revolution may now be derived from the nonlinear equations of equilibrium. In Eqs. (6), we let

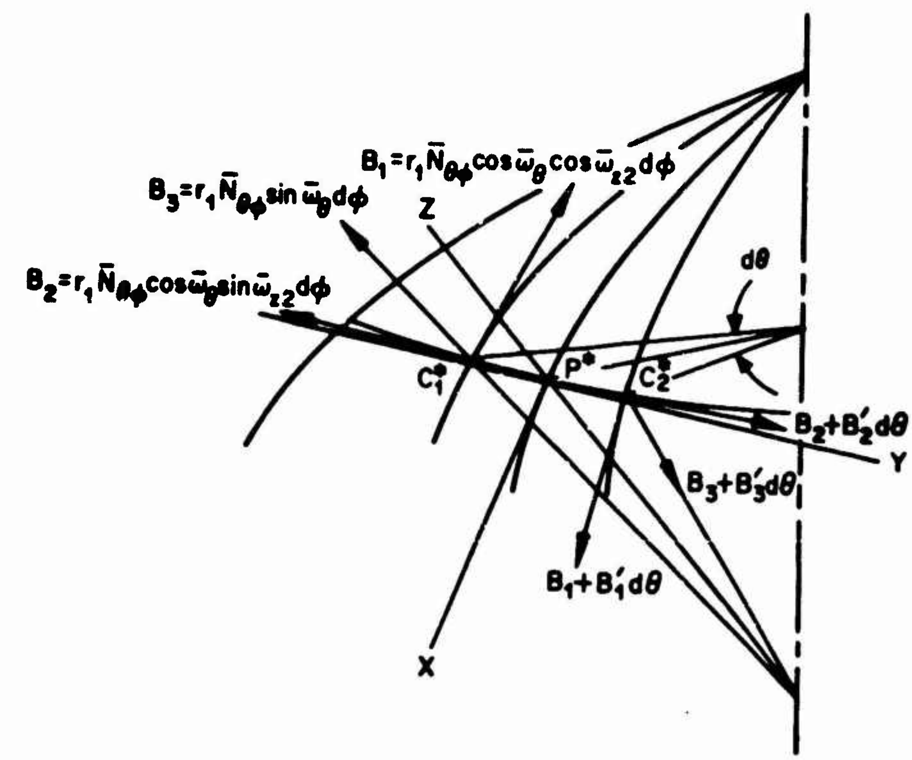
$$\overline{N}_\phi = N_{\phi 0} + N_\phi \,, \ldots ,\ \overline{M}_{\theta\phi} = M_{\theta\phi 0} + M_{\theta\phi} \,, \ldots , \tag{7a}$$

$$\bar{\omega}_\phi = \omega_{\phi 0} + \omega_\phi \,, \ldots , \quad . \tag{7b}$$

where, for example, $N_{\phi 0}$ is a prebuckling quantity and N_ϕ is the increment in this quantity which develops during buckling. After insertion of Eqs. (7) into Eqs. (6), the terms containing only prebuckling quantities may be subtracted



(a)



(b)

Fig. 8 Components of $\bar{N}_{\theta\phi}$

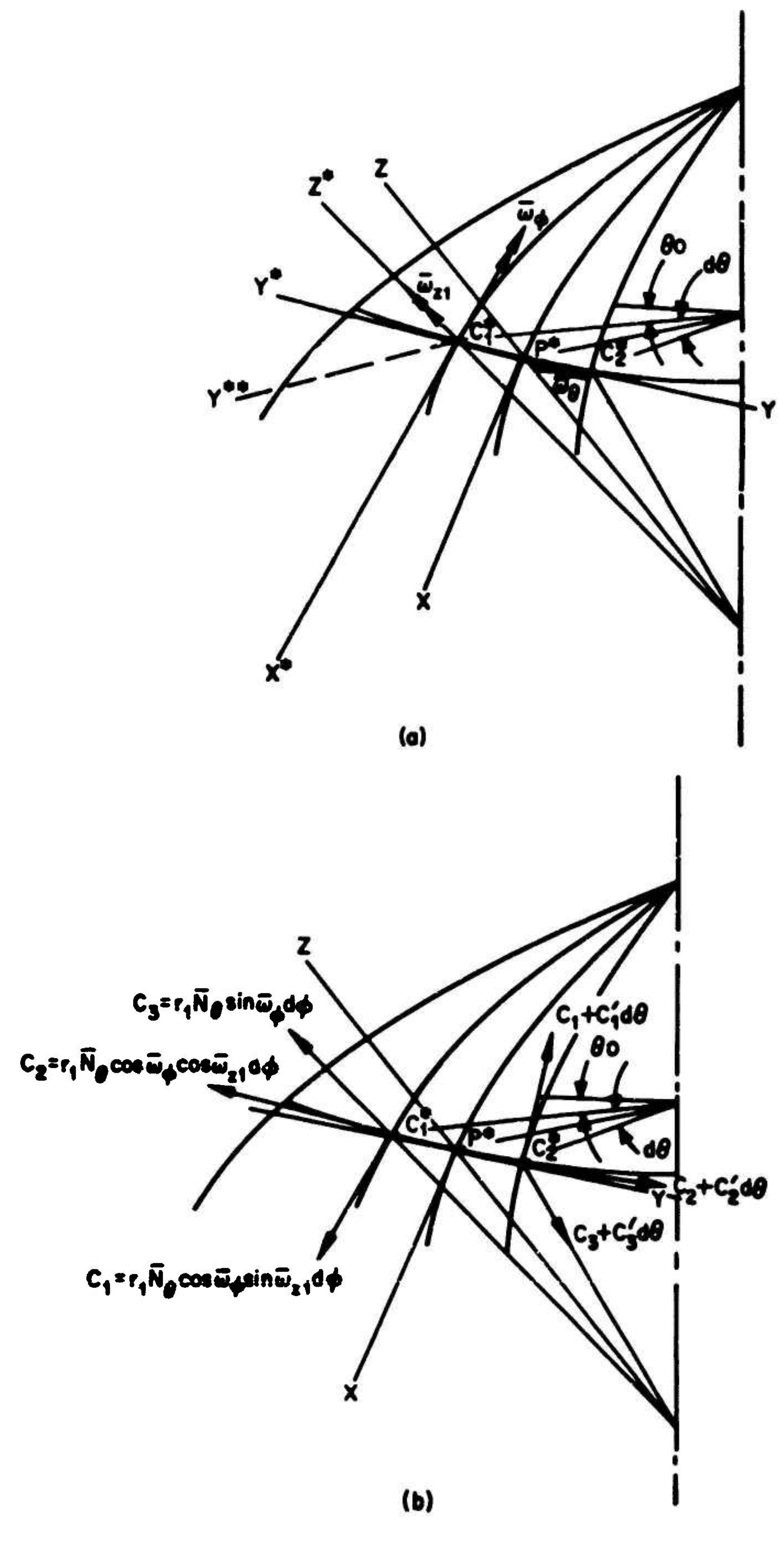


Fig. 9 Components of $\bar{N}_\theta$

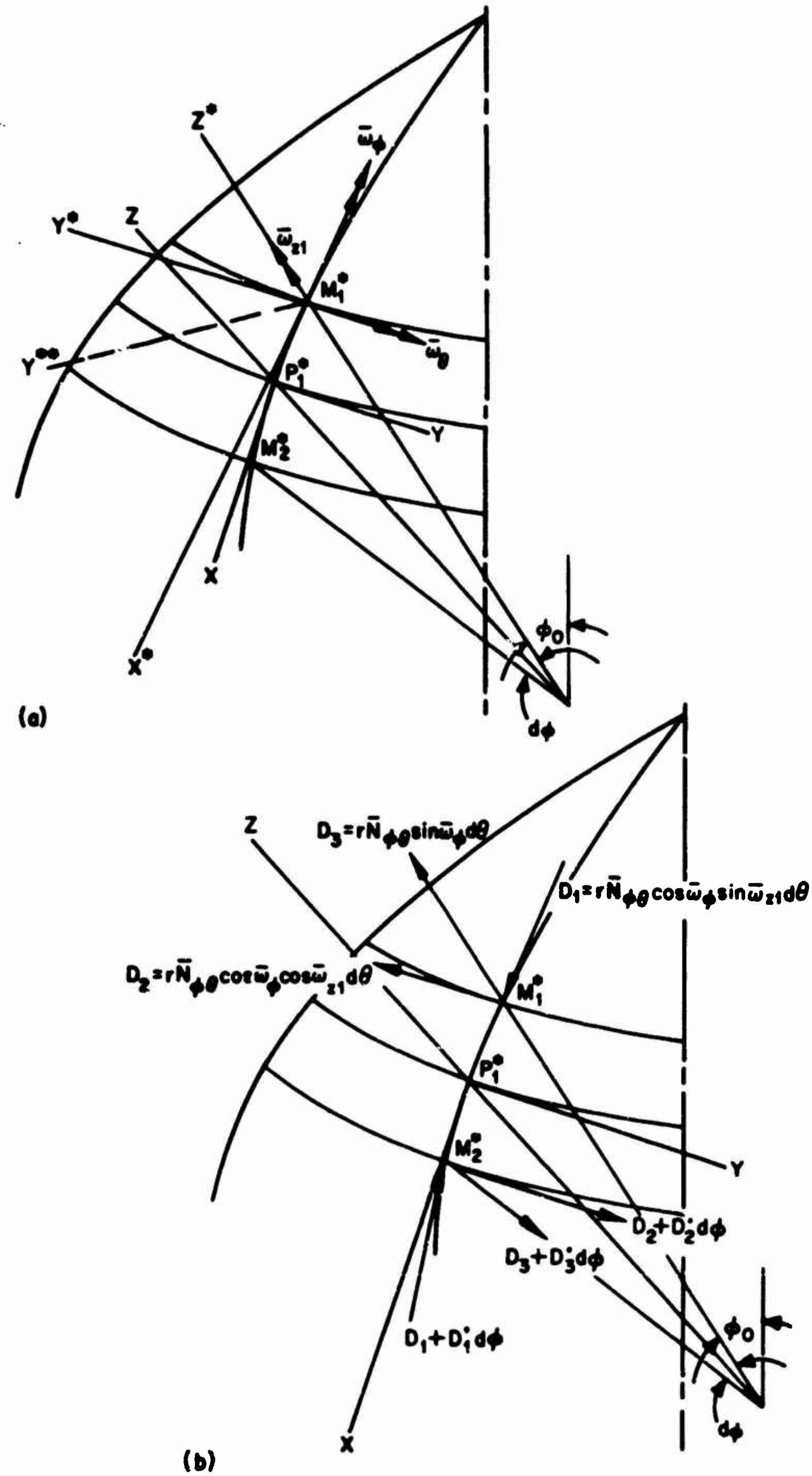


Fig. 10 Components of $\bar{N}_{\phi\theta}$

out by virtue of prebuckling equilibrium, and nonlinear terms in the incremental quantities may be neglected since the incremental quantities are considered to be infinitesimal. This procedure results in the replacement of the nonlinear equations of equilibrium by linear equations of stability. These equations are

$$\Sigma F_1 = (rN_\varphi)^{\cdot} + (r_1N_{\theta\varphi})' - r_1N_\theta \cos\varphi - rQ_\varphi - r(N_{\varphi 0}\omega_\theta + N_\varphi\omega_{\theta 0})$$
$$- r(N_{\varphi\theta 0}\omega_\varphi + N_{\varphi\theta}\omega_{\varphi 0}) - (rN_{\varphi 0 0}\omega_{z1} + rN_{\varphi 0}\omega_{z10})^{\cdot}$$
$$- (r_1N_{\theta 0}\omega_{z1} + r_1N_\theta\omega_{z10})'$$
$$- r_1(N_{\theta\varphi 0}\omega_{z2} + N_{\theta\varphi}\omega_{z20})\cos\varphi + rr_1\delta_{ph}p_z\omega_\theta = 0 \qquad (8a)$$

$$\Sigma F_2 = (r_1N_\theta)' + (rN_{\varphi\theta})^{\cdot} + r_1N_{\theta\varphi}\cos\varphi - r_1Q_\theta \sin\varphi$$
$$- r_1(N_{\theta\varphi 0}\omega_\theta + N_{\theta\varphi}\omega_{\theta 0})\sin\varphi - r_1(N_{\theta 0}\omega_\varphi + N_\theta\omega_{\varphi 0})\sin\varphi$$
$$+ (r_1N_{\theta\varphi 0}\omega_{z2} + r_1N_{\theta\varphi}\omega_{z20})' + (rN_{\varphi 0}\omega_{z2} + rN_\varphi\omega_{z20})^{\cdot}$$
$$- r_1(N_{\theta 0}\omega_{z1} + N_\theta\omega_{z10})\cos\varphi + rr_1\delta_{ph}p_z\omega_\varphi = 0 \qquad (8b)$$

$$\Sigma F_3 = -rN_\varphi - r_1N_\theta \sin\varphi - (rQ_\varphi)^{\cdot} - (r_1Q_\theta)' - (rN_{\varphi 0}\omega_\theta + rN_\varphi\omega_{\theta 0})^{\cdot}$$
$$- (r_1N_{\theta\varphi 0}\omega_\theta + r_1N_{\theta\varphi}\omega_{\theta 0})' - (rN_{\varphi\theta 0}\omega_\varphi + rN_{\varphi\theta}\omega_{\varphi 0})^{\cdot}$$
$$- (r_1N_{\theta 0}\omega_\varphi + r_1N_\theta\omega_{\varphi 0})' + r(N_{\varphi\theta 0}\omega_{z1} + N_{\varphi\theta}\omega_{z10})$$
$$- r_1(N_{\theta\varphi 0}\omega_{z2} + N_{\theta\varphi}\omega_{z20})\sin\varphi + rr_1\delta_{ph}p_z(\bar{\epsilon}_\varphi + \bar{\epsilon}_\theta) = 0 \qquad (8c)$$

$$\Sigma M_1 = (rM_{\varphi\theta})^{\cdot} + r_1M_\theta' + r_1M_{\theta\varphi}\cos\varphi - rr_1Q_\theta = 0 \qquad (8d)$$

$$\Sigma M_2 = -(rM_\varphi)^{\cdot} - rM_{\theta\varphi}' + r_1M_\theta\cos\varphi + rr_1Q_\varphi = 0 \qquad (8e)$$

When prebuckling rotations are neglected, these equations reduce to the previously derived equations of stability [Eqs. (5)].

Nonlinear equations of equilibrium for a general shell have been derived by Sanders (Ref. 23) and Kempner (Ref. 24). In Ref. 23, the derivation was carried out in tensor form whereas Ref. 24 used a variational approach. The results

of the present analysis, Eqs. (6), agree with those of Refs. 23 and 24 except for terms which involve rotations around the normal. Kempner did not include such terms whereas Sanders used an average rotation around the normal given by $\omega_z = \omega_{z1} + \omega_{z2}$. The present analysis uses ω_{z1} and ω_{z2} for the rotations about the normal of the tangent to a parallel circle and the tangent to the meridian. Thus, the discrepancy between the present results and those of Sanders is due, in part, to the use of different expressions for the rotation around the normal.

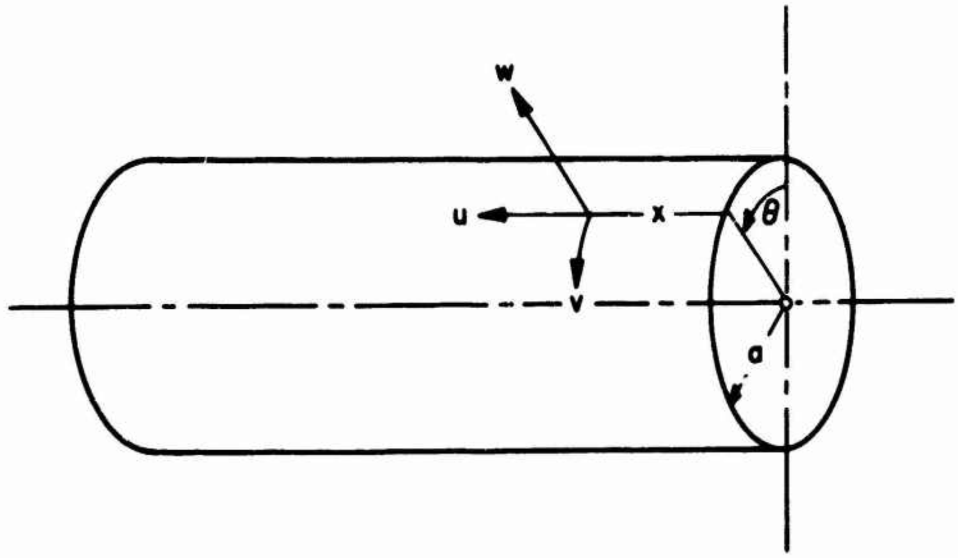
3. Equations of Equilibrium for a Cylinder

The equilibrium equations (5) will now be specialized for a circular cylindrical shell subjected to a uniform external pressure p, and an axial compression at the edges. The axial force per unit length of circumference is denoted by P. The sign conventions for the coordinates (x, θ), displacements (u, v, w), and stress resultants are shown in Fig. 11 (compare with Figs. III-2 and III-3). Thus, in the equilibrium equations, we let $\phi \to \frac{\pi}{2}$, $r_1 d\phi \to dx$, $r_1 \to \infty$, $r_2 \to a$, and $r \to a$. Also, we let $\omega_\theta = \frac{-\partial w}{\partial x}$, $\omega_\phi = -\frac{1}{a}(\frac{\partial w}{\partial \theta} - v)$, $\omega_{z1} = \frac{-1}{a}\frac{\partial u}{\partial \theta}$, $\omega_{z2} = \frac{\partial v}{\partial x}$, $\bar{\epsilon}_\theta = \frac{w}{a} + \frac{1}{a}\frac{\partial v}{\partial \theta}$, $\bar{\epsilon}_\phi = \frac{\partial u}{\partial x}$, $N_{\phi 0} = -P$, $N_{\theta 0} = -pa$, $N_{\phi\theta 0} = 0$, and $\delta_{ph} = 1$ (assume p is a hydrostatic pressure). This yields a set of equilibrium equations which are identical to those derived by Flügge in 1932 (Ref. 2):

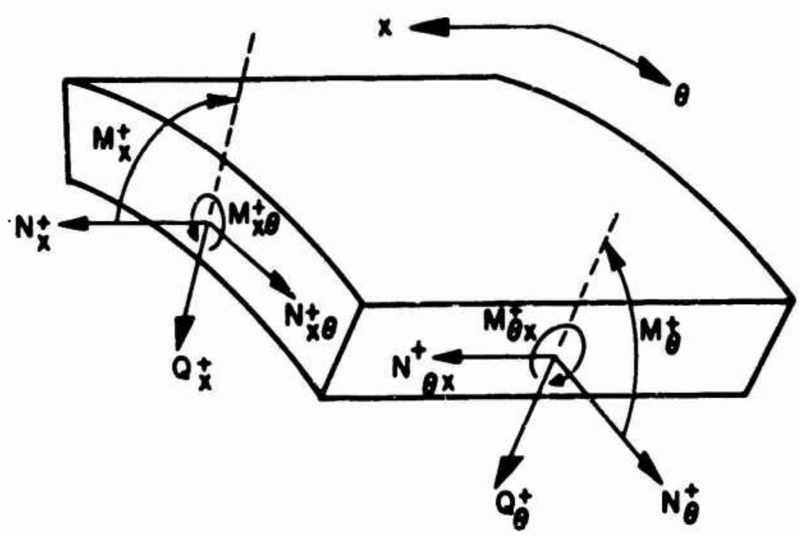
$$\frac{\partial}{\partial x}N_x + \frac{1}{a}\frac{\partial}{\partial \theta}N_{\theta x} + p\left(\frac{\partial w}{\partial x} - \frac{1}{a}\frac{\partial^2 u}{\partial \theta^2}\right) = 0 \tag{19a}$$

$$\frac{\partial}{\partial \theta}N_\theta + a\frac{\partial}{\partial x}N_{x\theta} - \frac{1}{a}\frac{\partial}{\partial \theta}M_\theta - \frac{\partial}{\partial x}M_{x\theta} - Pa\frac{\partial^2 v}{\partial x^2} = 0 \tag{19b}$$

$$\frac{1}{a}\frac{\partial^2}{\partial \theta^2}M_\theta + 2\frac{\partial^2}{\partial x \partial \theta}M_{x\theta} + a\frac{\partial^2}{\partial x^2}M_x + N_\theta$$
$$+ pa\left(\frac{\partial u}{\partial x} + \frac{w}{a} + \frac{1}{a}\cdot\frac{\partial^2 w}{\partial \theta^2}\right) + Pa\frac{\partial^2 w}{\partial x^2} = 0 \ . \tag{19c}$$



(a) COORDINATES & DISPLACEMENTS



(b) STRESS RESULTANTS

Fig. 11 Conventions for Coordinates, Displacements, and Stress Resultants of a Cylinder

4. Equations of Equilibrium for a Sphere

The equations of equilibrium for axially symmetric buckling of a sphere loaded by a uniform hydrostatic pressure p are obtained from Eqs. (5) by letting $r_1 = a$ (radius of sphere), $r = a \sin \varphi$, $\omega_\theta = -\frac{1}{a}(w^{\cdot} - v)$, $\bar{\epsilon}_\theta = \frac{1}{a \sin \varphi}(v \cos \varphi + w \sin \varphi)$, $\bar{\epsilon}_\varphi = \frac{1}{a}(w + v^{\cdot})$, $N_{\varphi 0} = -p\frac{a}{2}$, $N_{\theta 0} = -p\frac{a}{2}$, and $\frac{\partial}{\partial \theta}(\ ) = u = N_{\varphi\theta 0} = \omega_{z1} = \omega_{z2} = \omega_\varphi = 0$. This yields a set of equilibrium equations which are identical to those derived by Flügge (Ref. 13):

$$(N_\varphi \sin \varphi)^{\cdot} - N_\theta \cos \varphi - Q_\varphi \sin \varphi + \frac{p}{2}(w^{\cdot} - v) \sin \varphi = 0 \tag{10a}$$

$$\begin{aligned}(N_\varphi + N_\theta) \sin \varphi + (Q_\varphi \sin \varphi)^{\cdot} + \frac{p}{2}(v^{\cdot} \sin \varphi + v \cos \varphi \\ + 4\, w \sin \varphi + \dot{w} \cos \varphi + \ddot{w} \sin \varphi) = 0\end{aligned} \tag{10b}$$

$$(M_\varphi \sin \varphi)^{\cdot} - M_\theta \cos \varphi - a\, Q_\varphi \sin \varphi = 0 \; . \tag{10c}$$

V

SHELLS OF REVOLUTION UNDER AXIALLY SYMMETRIC LOADS

In this chapter, and in the sequel, only axisymmetrically loaded shells of revolution are considered. Also, the effects of prebuckling rotations are neglected. Since the loading is axially symmetric, the space variables in the partial differential equations governing the stability of a shell of revolution can be separated. After the separation of variables, three equations in terms of the three displacement components are obtained through combination of the equilibrium equations and the elastic law.

1. Separation of Variables

The stability equations for a shell of revolution are given by a system of partial differential equations with variable coefficients. For the case of axially symmetric loading, the coefficients in these equations are independent of the circumferential coordinate θ (see Fig. III-1 for notation). Consequently, it is possible to separate variables and thus replace the system of partial differential equations by a system of ordinary differential equations. Such a separation of variables is effected by means of the following Fourier series representation for the incremental quantities:

$$v = \sum_{n=0}^{\infty} v_n(\phi) \cos n\theta$$

$$w = \sum_{n=0}^{\infty} w_n(\phi) \cos n\theta$$

$$N_\phi = \sum_{n=0}^{\infty} N_{\phi n}(\phi) \cos n\theta$$

$$N_\theta = \sum_{n=0}^{\infty} N_{\theta n}(\phi) \cos n\theta$$

$$M_\phi = \sum_{n=0}^{\infty} M_{\phi n}(\phi) \cos n\theta$$

$$M_\theta = \sum_{n=0}^{\infty} M_{\theta n}(\phi) \cos n\theta$$

$$Q_\phi = \sum_{n=0}^{\infty} Q_{\phi n}(\phi) \cos n\theta$$

$$\omega_\theta = \sum_{n=0}^{\infty} \omega_{\theta n}(\phi) \cos n\theta$$

$$\bar{\epsilon}_\phi = \sum_{n=0}^{\infty} \bar{\epsilon}_{\phi n}(\phi) \cos n\theta$$

$$\bar{\epsilon}_\theta = \sum_{n=0}^{\infty} \bar{\epsilon}_{\theta n}(\phi) \cos n\theta \quad .$$

$$u = \sum_{n=1}^{\infty} u_n(\phi) \sin n\theta$$

$$N_{\phi\theta} = N_{\theta\phi} = \sum_{n=1}^{\infty} N_{\phi\theta n}(\phi) \sin n\theta$$

$$M_{\phi\theta} = M_{\theta\phi} = \sum_{n=1}^{\infty} M_{\phi\theta n}(\phi) \sin n\theta$$

$$Q_\theta = \sum_{n=1}^{\infty} Q_{\theta n}(\phi) \sin n\theta$$

$$\omega_\phi = \sum_{n=1}^{\infty} \omega_{\phi n}(\phi) \sin n\theta$$

$$\omega_{z1} = \sum_{n=1}^{\infty} \omega_{z1n}(\phi) \sin n\theta \qquad (1)$$

$$\omega_{z2} = \sum_{n=1}^{\infty} \omega_{z2n}(\phi) \sin n\theta$$

These expressions represent the general solution of the stability equations for the cases in which

(i) the shell is complete in the circumferential direction, i.e., $0 \leq \theta \leq 2\pi$

(ii) the boundary conditions on the edges $\theta = -\frac{\theta_0}{2}$, $\frac{\theta_0}{2}$ for a partial toroidal shell are

$$w = M_\theta = v = N_\theta = 0 \quad . \tag{2}$$

For the first case, the requirement that the incremental quantities should be periodic functions of θ is satisfied by each term of the Fourier series in Eqs. (1). For the second case, the boundary conditions [Eq. (2)] are satisfied by each term of the Fourier series obtained by assigning values to n in Eqs. (1) as follows:

$$n = n^* \frac{\pi}{\theta_0} \ , \ n^* = 1, 3, 5, \ldots \quad . \tag{3}$$

2. Equilibrium Equations

Insertion of the Fourier series representations of the incremental quantities [Eqs. (1)] into the equations of equilibrium [Eqs. (IV-5)] results in the following equations of equilibrium for each value of n :

$$\begin{aligned} \Sigma F_{1n} = {} & (rN_{\phi n})^{\cdot} + nr_1 N_{\theta\phi n} - r_1 N_{\theta n} \cos\phi - rQ_{\phi n} - rN_{\phi 0}\omega_{\theta n} \\ & - nr_1 N_{\theta 0}\omega_{z1n} + rr_1 \delta_{ph} p_z \omega_{\theta n} = 0 \end{aligned} \tag{4a}$$

$$\begin{aligned} \Sigma F_{2n} = {} & (rN_{\phi\theta n})^{\cdot} - nr_1 N_{\theta n} + r_1 N_{\theta\phi n}\cos\phi - r_1 Q_{\theta n} \sin\phi \\ & - r_1 N_{\theta 0}\omega_{\phi n} \sin\phi - r_1 N_{\theta 0}\omega_{z1n} \cos\phi \\ & + (r_1 N_{\phi 0} \cos\phi + r\dot{N}_{\phi 0})\,\omega_{z2n} + (rN_{\phi 0})\,\dot{\omega}_{z2n} \\ & + rr_1 \delta_{ph} p_z \omega_{\phi n} = 0 \end{aligned} \tag{4b}$$

$$\Sigma F_{3n} = rN_{\phi n} + r_1 N_{\theta n} \sin\phi + nr_1 Q_{\theta n} + (rQ_{\phi n})$$

$$+ (r_1 N_{\phi 0} \cos\phi + r\dot{N}_{\phi 0})\,\omega_{\theta n} + rN_{\phi 0}\,\dot{\omega}_{\theta n}$$

$$+ nr_1 N_{\theta 0}\,\omega_{\phi n} - rr_1 \delta_{ph} p_z (\bar{\epsilon}_{\phi n} + \bar{\epsilon}_{\theta n}) = 0 \qquad (4c)$$

$$\Sigma M_{1n} = (rM_{\phi\theta n})^{\cdot} - nr_1 M_{\theta n} + r_1 M_{\phi\theta n} \cos\phi - rr_1 Q_{\theta n} = 0 \qquad (4d)$$

$$\Sigma M_{2n} = (rM_{\phi n})^{\cdot} + nr_1 M_{\phi\theta n} - r_1 M_{\theta n} \cos\phi - rr_1 Q_{\phi n} = 0 \qquad (4e)$$

3. Elastic Law

The following relations, which are obtained from the elastic law [Eqs. (III-18)], can be used to express the equilibrium equations in terms of the displacement components u_n, v_n, and w_n

$$\frac{1}{D}(rN_{\phi n}) = (\nu n)\,u_n + (\nu y_1)\,v_n + \left(\frac{r}{r_1}\right)\dot{v}_n + \left(\frac{r}{r_1} + \nu x_1\right) w_n \qquad (5a)$$

$$\frac{1}{D}(r_1 N_{\theta n}) = \left(\frac{nr_1}{r}\right) u_n + \left(\frac{r_1 y_1}{r}\right) v_n + \nu\dot{v}_n + \left(\frac{r_1 x_1}{r} + \nu\right) w_n \qquad (5b)$$

$$\frac{1}{D}(r_1 N_{\theta\phi n}) = \left[-\left(\frac{1-\nu}{2}\right)\frac{r_1 y_1}{r}\right] u_n + \left(\frac{1-\nu}{2}\right)\dot{u}_n + \left[-\left(\frac{1-\nu}{2}\right) n\,\frac{r_1}{r}\right] v_n \qquad (5c)$$

$$\frac{1}{K}(r_1 M_{\theta n}) = \left(-n\,\frac{r_1 x_1}{r^2}\right) u_n + \left(-\frac{y_1}{r} + \frac{\nu \dot{r}_1}{r_1^2}\right) v_n + \left(-\nu\,\frac{1}{r_1}\right)\dot{v}_n$$

$$+ \left(-n^2\,\frac{r_1}{r^2}\right) w_n + \left(\frac{y_1}{r} - \frac{\nu \dot{r}_1}{r_1^2}\right)\dot{w}_n + \left(\nu\,\frac{1}{r_1}\right)\ddot{w}_n \qquad (5d)$$

$$\frac{1}{K}(r_1 M_{\theta\phi n}) = \left[-\left(\frac{1-\nu}{2}\right)\frac{y_1}{r} + (1-\nu)\,\frac{r_1 x_1 y_1}{r^2}\right] u_n + \left[-\left(\frac{1-\nu}{2}\right)\frac{x_1}{r}\right]\dot{u}_n$$

$$+ \left[\left(\frac{1-\nu}{2}\right) n\,\frac{1}{r}\right] v_n + \left[(1-\nu)\,n\,\frac{r_1 y_1}{r^2}\right] w_n + \left[-(1-\nu)\,n\,\frac{1}{r}\right]\dot{w}_n$$

$$(5e)$$

where, for brevity, we have introduced the notation:

$$x_1 = \sin\phi \ ; \ y_1 = \cos\phi \ ; \ \text{and} \ y_2 = \cos 2\phi \tag{6}$$

4. Rotation and Strain Components

By substitution of Eqs. (1) into Eqs. (III-8), (III-9), (III-11), and (III-17), we obtain the following expressions for the rotation and strain components:

$$\omega_{\theta n} = -\frac{1}{r_1}\dot{w}_n + \frac{1}{r_1}v_n \tag{7a}$$

$$\omega_{\phi n} = \frac{n}{r}w_n + \frac{x_1}{r}u_n \tag{7b}$$

$$\omega_{z1n} = \frac{n}{r}v_n \tag{7c}$$

$$\omega_{z2n} = \frac{1}{r_1}\dot{u}_n - \frac{y_1}{r}u_n \tag{7d}$$

$$\bar{\epsilon}_{\phi n} = \frac{\dot{v}_n}{r_1} + \frac{w_n}{r_1} \tag{7e}$$

$$\bar{\epsilon}_{\theta n} = \frac{n}{r}u_n + \frac{y_1}{r}v_n + \frac{x_1}{r}w_n \tag{7f}$$

5. Stability Equations for the Axially Symmetric Loaded Shell of Revolution

For a thin shell, the effect of the transverse shear forces $Q_{\phi n}$ and $Q_{\theta n}$ in the first two equations of equilibrium [Eqs. (4)] may be omitted. [Also we note that Steele (Ref. 25) has pointed out that the retention of these terms in the first two equations of equilibrium is inconsistent with the use of an uncoupled elastic law.] The moment equations of equilibrium [Eqs. (4d–e)] may be used to eliminate $Q_{\phi n}$ and $Q_{\theta n}$ from Eq. (4c), and the stress resultants, rotations, and

strains may be eliminated from the equilibrium equations through insertion of Eqs. (5 and 7) into Eqs. (4). In this way, we finally obtain a set of three ordinary differential equations for the three displacement components $u_n(\phi)$, $v_n(\phi)$, and $w_n(\phi)$:

$$(h_1 + c_1) u_n + (h_2 + c_2) \dot{u}_n + (h_3 + c_3) \ddot{u}_n + (h_4 + c_4) v_n + (h_5 + c_5) \dot{v}_n$$
$$+ (h_6 + c_6) \ddot{v}_n + (h_7 + c_7) \dddot{v}_n + (h_8 + c_8) w_n + (h_9 + c_9) \dot{w}_n$$
$$+ (h_{10} + c_{10}) \ddot{w}_n + (h_{11} + c_{11}) \dddot{w}_n + (h_{12} + c_{12}) \ddddot{w}_n = 0 \tag{8a}$$

$$(f_1 + a_1) u_n + (f_2 + a_2) \dot{u}_n + (f_3 + a_3) \ddot{u}_n + (f_4 + a_4) v_n + (f_5 + a_5) \dot{v}_n$$
$$+ (f_6 + a_6) \ddot{v}_n + (f_7 + a_7) \dddot{v}_n + (f_8 + a_8) w_n + (f_9 + a_9) \dot{w}_n$$
$$+ (f_{10} + a_{10}) \ddot{w}_n + (f_{11} + a_{11}) \dddot{w}_n + (f_{12} + a_{12}) \ddddot{w}_n = 0 \tag{8b}$$

$$(g_1 + b_1) u_n + (g_2 + b_2) \dot{u}_n + (g_3 + b_3) \ddot{u}_n + (g_4 + b_4) v_n + (g_5 + b_5) \dot{v}_n$$
$$+ (g_6 + b_6) \ddot{v}_n + (g_7 + b_7) \dddot{v}_n + (g_8 + b_8) w_n + (g_9 + b_9) \dot{w}_n$$
$$+ (g_{10} + b_{10}) \ddot{w}_n + (g_{11} + b_{11}) \dddot{w}_n + (g_{12} + b_{12}) \ddddot{w}_n = 0 \tag{8c}$$

where the coefficients a , b , c , f , g , h , which depend on ϕ , are given by the following formulas:

$$h_1 = K\left[n \frac{x_1}{rr_1} + n \frac{\dot{r}_1 y_1}{rr_1^2} + (1 + \nu)\, n \frac{y_1^2}{r^2} + (2 - \nu)\, n \frac{y_2}{r^2} - n \frac{x_1^2}{r^2} - 4n \frac{r_1 x_1 y_1^2}{r^3} + n^3 \frac{r_1 x_1}{r^3}\right] + D\left[\nu n + n \frac{r_1 x_1}{r}\right] \tag{9a}$$

$$h_2 = K\left[-2n \frac{y_1}{rr_1} + n \frac{\dot{r}_1 x_1}{rr_1^2} + 3n \frac{x_1 y_1}{r^2}\right] \tag{9b}$$

$$h_3 = K\left[-n\frac{x_1}{rr_1}\right] \tag{9c}$$

$$h_4 = K\left[-2\frac{x_1y_1}{rr_1} - n^2\frac{\dot{r}_1}{rr_1^2} - \frac{\dot{r}_1y_1^2}{rr_1^2} + n^2\frac{y_1}{r^2} - \frac{y_1^3}{r^2} + \frac{\nu y_1}{r_1^2} + 2\frac{\ddot{r}_1y_1}{r_1^3}\right.$$

$$\left. - (1 + 2\nu)\frac{\dot{r}_1x_1}{r_1^3} - 6\frac{(\dot{r}_1)^2y_1}{r_1^4} + \frac{r\dddot{r}_1}{r_1^4} - 10\frac{r\dot{r}_1\ddot{r}_1}{r_1^5} + 15\frac{r(\dot{r}_1)^3}{r_1^6}\right]$$

$$+ D\left[\nu y_1 + \frac{r_1x_1y_1}{r}\right] \tag{9d}$$

$$h_5 = K\left[+n^2\frac{1}{rr_1} + \frac{y_1^2}{rr_1} + (1 + \nu)\frac{x_1}{r_1^2} + 6\frac{\dot{r}_1y_1}{r_1^3} + 4\frac{r\ddot{r}_1}{r_1^4} - 15\frac{r(\dot{r}_1)^2}{r_1^5}\right]$$

$$+ D\left[\frac{r}{r_1} + \nu x_1\right] \tag{9e}$$

$$h_6 = K\left[-2\frac{y_1}{r_1^2} + 6\frac{r\dot{r}_1}{r_1^4}\right] \tag{9f}$$

$$h_7 = K\left[-\frac{r}{r_1^3}\right] \tag{9g}$$

$$h_8 = K\left[-(3 - \nu)n^2\frac{x_1}{r^2} - 4n^2\frac{r_1y_1^2}{r^3} + n^4\frac{r_1}{r^3}\right] + D\left[\frac{r}{r_1} + r_1\frac{x_1^2}{r} + 2\nu x_1\right] \tag{9h}$$

$$h_9 = K\left[+2\frac{x_1y_1}{rr_1} + 2n^2\frac{\dot{r}_1}{rr_1^2} + \frac{\dot{r}_1y_1^2}{rr_1^2} + 2n^2\frac{y_1}{r^2} + \frac{y_1^3}{r^2} - \frac{\nu y_1}{r_1^2} + (1 + 2\nu)\frac{\dot{r}_1x_1}{r_1^3}\right.$$

$$\left. - 2\frac{\ddot{r}_1y_1}{r_1^3} + 6\frac{(\dot{r}_1)^2y_1}{r_1^4} - \frac{r\dddot{r}_1}{r_1^4} + 10\frac{r\dot{r}_1\ddot{r}_1}{r_1^5} - 15\frac{r(\dot{r}_1)^3}{r_1^6}\right] \tag{9i}$$

$$h_{10} = K\left[-2n^2\frac{1}{rr_1} - \frac{y_1^2}{rr_1} - (1 + \nu)\frac{x_1}{r_1^2} - 6\frac{\dot{r}_1y_1}{r_1^3} - 4\frac{r\ddot{r}_1}{r_1^4} + 15\frac{r(\dot{r}_1)^2}{r_1^5}\right] \tag{9j}$$

$$h_{11} = K\left[[\text{illegible}]\,\frac{y_1}{r_1^{\,2}} - 6\,\frac{r\dot{r}_1}{r_1^{\,4}} \right] \tag{9k}$$

$$h_{12} = K\left[\frac{r}{r_1^{\,3}} \right] \tag{9l}$$

$$f_1 = D\left[-\frac{1}{2}(3-\nu)n\,\frac{r_1 y_1}{r} \right] \tag{9m}$$

$$f_2 = D\left[\left(\frac{1+\nu}{2}\right) n \right] \tag{9n}$$

$$f_3 = 0 \tag{9o}$$

$$f_4 = D\left[-\nu x_1 - \left(\frac{1-\nu}{2}\right) n^2 \frac{r_1}{r} - \frac{r_1 y_1^{\,2}}{r} \right] \tag{9p}$$

$$f_5 = D\left[y_1 - \frac{r\dot{r}_1}{r_1^{\,2}} \right] \tag{9q}$$

$$f_6 = D\left[\frac{r}{r_1} \right] \tag{9r}$$

$$f_7 = 0 \tag{9s}$$

$$f_8 = D\left[y_1 - \frac{r_1 x_1 y_1}{r} - \frac{r\dot{r}_1}{r_1^{\,2}} \right] \tag{9t}$$

$$f_9 = D\left[\frac{r}{r_1} + \nu x_1 \right] \tag{9u}$$

$$f_{10} = f_{11} = f_{12} = 0 \tag{9v}$$

$$g_1 = D\left[\left(\frac{1-\nu}{2}\right) x_1 - n^2 \frac{r_1}{r} - \left(\frac{1-\nu}{2}\right) \frac{r_1 y_1^{\,2}}{r} \right] \tag{9w}$$

$$g_2 = D\left[-\left(\frac{1-\nu}{2}\right) \frac{r\dot{r}_1}{r_1^{\,2}} + \left(\frac{1-\nu}{2}\right) y_1 \right] \tag{9x}$$

$$g_3 = D\left[\left(\frac{1-\nu}{2}\right)\frac{r}{r_1}\right] \tag{9y}$$

$$g_4 = D\left[-\left(\frac{3-\nu}{2}\right)n\,\frac{r_1 y_1}{r}\right] \tag{9z}$$

$$g_5 = D\left[-\left(\frac{1+\nu}{2}\right)n\right] \tag{9a'}$$

$$g_6 = 0 \tag{9b'}$$

$$g_7 = 0 \tag{9c'}$$

$$g_8 = D\left[-\nu n - n\,\frac{r_1 x_1}{r}\right] \tag{9d'}$$

$$g_9 = 0 \tag{9e'}$$

$$g_{10} = g_{11} = g_{12} = 0 \tag{9f'}$$

$$c_1 = n\frac{r_1 x_1}{r} N_{\theta 0} - n r_1 \delta_{ph} p_z \tag{9g'}$$

$$c_2 = c_3 = 0 \tag{9h'}$$

$$c_4 = y_1 N_{\phi 0} + \frac{r}{r_1}\dot{N}_{\phi 0} - \frac{r\dot{r}_1}{r_1^2} N_{\phi 0} - r_1 y_1 \delta_{ph} p_z \tag{9i'}$$

$$c_5 = \frac{r}{r_1} N_{\phi 0} - r\delta_{ph} p_z \tag{9j'}$$

$$c_6 = c_7 = 0 \tag{9k'}$$

$$c_8 = n^2 \frac{r_1}{r} N_{\theta 0} - r_1 x_1 \delta_{ph} p_z - r\delta_{ph} p_z \tag{9l'}$$

$$c_9 = -y_1 N_{\phi 0} - \frac{r}{r_1}\dot{N}_{\phi 0} + r\frac{\dot{r}_1}{r_1^2} N_{\phi 0} \tag{9m'}$$

$$c_{10} = -\frac{r}{r_1} N_{\varphi 0} \tag{9n'}$$

$$c_{11} = c_{12} = 0 \tag{9o'}$$

$$a_1 = a_2 = a_3 = 0 \tag{9p'}$$

$$a_4 = -\frac{r}{r_1} N_{\varphi 0} + r\delta_{ph} p_z - n^2 \frac{r_1}{r} N_{\theta 0} \tag{9q'}$$

$$a_5 = a_6 = a_7 = a_8 = 0 \tag{9r'}$$

$$a_9 = \frac{r}{r_1} N_{\varphi 0} - r\delta_{ph} p_z \tag{9s'}$$

$$a_{10} = a_{11} = a_{12} = 0 \tag{9t'}$$

$$b_1 = -\frac{r_1 x_1^2}{r} N_{\theta 0} + r_1 x_1 \delta_{ph} p_z + x_1 N_{\varphi 0} - y_1 \dot{N}_{\varphi 0} \tag{9u'}$$

$$b_2 = -\frac{r \dot{r}_1}{r_1^2} N_{\varphi 0} + \frac{r}{r_1} \dot{N}_{\varphi 0} \tag{9v'}$$

$$b_3 = \frac{r}{r_1} N_{\varphi 0} \tag{9w'}$$

$$b_4 = -n \frac{r_1 y_1}{r} N_{\theta 0} \tag{9x'}$$

$$b_5 = b_6 = b_7 = 0 \tag{9y'}$$

$$b_8 = -n \frac{r_1 x_1}{r} N_{\theta 0} + n r_1 \delta_{ph} p_z \tag{9z'}$$

$$b_9 = b_{10} = b_{11} = b_{12} = 0 \tag{9a''}$$

For convenience, a subscript n was omitted in the expressions for the coefficients a , b , c , f , g , and h in Eqs. (8) and (9). We note that the functions $a(\phi)$, $b(\phi)$, and $c(\phi)$ in Eqs. (8) depend on the prebuckling quantities $N_{\phi 0}$, $N_{\theta 0}$, and p_z whereas $f(\phi)$, $g(\phi)$, $h(\phi)$ do not.

VI

TOROIDAL SHELL UNDER EXTERNAL PRESSURE—THEORETICAL ANALYSIS

1. Stability Equations for a Toroidal Shell Under External Pressure

As an application of the theory developed in the preceding chapters, the equations governing the stability of a general shell of revolution will now be specialized for a toroidal shell subject to a uniform hydrostatic pressure p .

The notation for a toroidal shell with a circular meridian is shown in Fig. 1. The radius of curvature of the meridian r_1 is denoted by a , and the distance between the center of the circular cross section and the axis of revolution is denoted by b . Note that the shell geometry and the applied loading are symmetric about the plane A-A (see Fig. 1). Hence, it is expected that the buckling pattern will be either symmetric or antimetric about this plane. Accordingly, it is convenient to use instead of ϕ the coordinate ψ measured from the plane A-A as shown in Fig. 1. The independent variable ψ is related to the colatitude ϕ used in Chapter V by

$$\phi = \psi + \frac{\pi}{2} \quad . \tag{1}$$

From Eqs. (1) and (III-4) we find

$$\frac{\partial}{\partial\psi}(\ ) = \frac{\partial}{\partial\phi}(\ ) = (\ )^{\bullet} \quad . \tag{2}$$

For brevity, we introduce the notations

$$S_m = \sin m\psi\ , \ m = 1,2,.. \tag{3a}$$

$$C_m = \cos m\psi\ , \ m = 0,1,2,... \tag{3b}$$

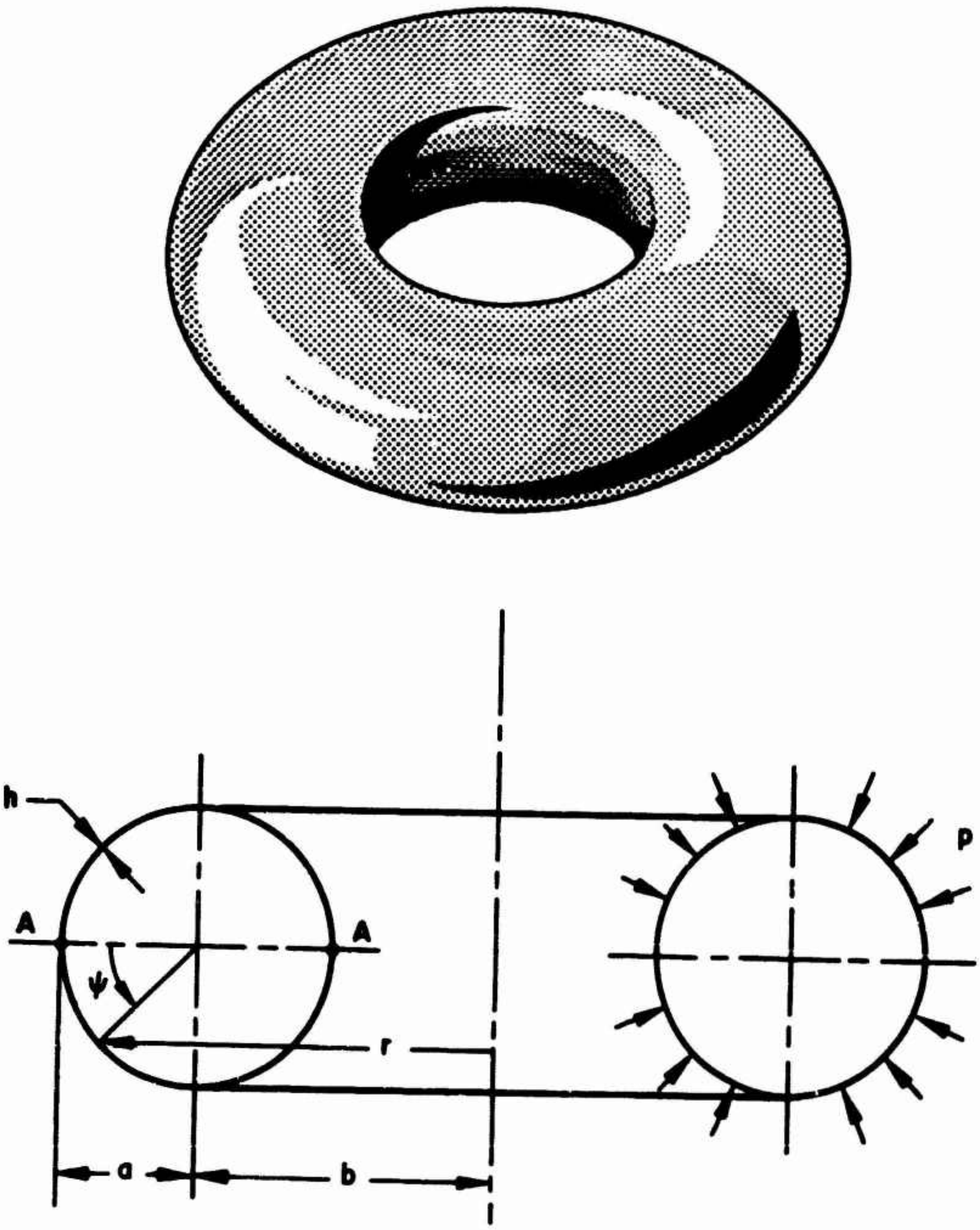


Fig. 1 Notation for a Toroidal Shell

Therefore, the notations $x_1 = \sin\phi$, $y_1 = \cos\phi$, and $y_2 = \cos 2\phi$ used in the preceding chapter are replaced by

$$x_1 = C_1 = \cos\psi \tag{4a}$$

$$y_1 = -S_1 = -\sin\psi \tag{4b}$$

$$y_2 = -C_2 = -\cos 2\psi \quad . \tag{4c}$$

From Fig. 1, we find that

$$r = b + a\cos\psi \quad ,$$

which can be rewritten as

$$\frac{r}{a} = \alpha + C_1 \tag{5}$$

where α is a nondemensional geometric parameter

$$\alpha = \frac{b}{a} \tag{6}$$

The prebuckling stress resultants $N_{\phi 0}$ and $N_{\theta 0}$ may be obtained from Ref. (13):

$$N_{\phi 0} = -\frac{pa}{b + a\sin\phi}\left(b + \frac{a}{2}\sin\phi\right)$$

$$N_{\theta 0} = -\frac{pa}{2} \quad .$$

We introduce a nondimensional load parameter

$$\lambda = \frac{pa}{Eh} \tag{7}$$

where E is Young's modulus and h is the shell thickness. Then the prebuckling stress resultants may be written as

$$\frac{1}{D} N_{\psi 0} = -\frac{1}{2}(1 - \nu^2)\frac{a}{r}(2\alpha + C_1)\lambda \tag{8a}$$

$$\frac{1}{D} N_{\theta 0} = -\frac{1}{2}(1 - \nu^2)\lambda \quad . \tag{8b}$$

where D is the extensional stiffness of the shell $\left[\text{see Eq. (III-19)}\right]$.

We may now obtain the stability equations for a toroidal shell through specialization of the equations for a general shell of revolution $\left[\text{Eqs. (V-8)}\right]$. We multiply Eq. (V-8a) by $\left(\frac{r}{a}\right)^3$, Eq. (V-8b) by $\frac{r}{a}$, and Eq. (V-8c) by $\left(\frac{r}{a}\right)^2$ in order to remove any dependence on ψ in the denominators of the coefficients in the stability equations. Next, we divide the resulting equations by D and introduce the nondimensional geometric parameter

$$k = \frac{K}{Da^2} = \frac{1}{12}\left(\frac{h}{a}\right)^2 \quad . \tag{9}$$

Finally, we substitute Eqs. (1) through (9) into Eqs. (V-8) and (V-9) and obtain the following stability equations for a toroidal shell subject to a uniform external pressure:

$$\begin{aligned}
&(h_1 + c_1\lambda)\, u_n(\psi) + (h_2 + c_2\lambda)\, \dot{u}_n(\psi) + (h_3 + c_3\lambda)\, \ddot{u}_n(\psi) \\
&\quad + (h_4 + c_4\lambda)\, v_n(\psi) + (h_5 + c_5\lambda)\, \dot{v}_n(\psi) \\
&\quad + (h_6 + c_6\lambda)\, \ddot{v}_n(\psi) + (h_7 + c_7\lambda)\, \dddot{v}_n(\psi) \\
&\quad + (h_8 + c_8\lambda)\, w_n(\psi) + (h_9 + c_9\lambda)\, \dot{w}_n(\psi) \\
&\quad + (h_{10} + c_{10}\lambda)\, \ddot{w}_n(\psi) + (h_{11} + c_{11}\lambda)\, \dddot{w}_n(\psi) \\
&\quad + (h_{12} + c_{12}\lambda)\, \ddddot{w}_n(\psi) = 0
\end{aligned} \tag{10a}$$

$$(f_1 + a_1\lambda)\, u_n(\psi) + (f_2 + a_2\lambda)\, \dot{u}_n(\psi) + (f_3 + a_3\lambda)\, \ddot{u}_n(\psi) + (f_4 + a_4\lambda)\, v_n(\psi)$$

$$+ (f_5 + a_5\lambda)\, \dot{v}_n(\psi) + (f_6 + a_6\lambda)\, \ddot{v}_n(\psi) + (f_7 + a_7\lambda)\, \dddot{v}_n(\psi)$$

$$+ (f_8 + a_8\lambda)\, w_n(\psi) + (f_9 + a_9\lambda)\, \dot{w}_n(\psi) + (f_{10} + a_{10}\lambda)\, \ddot{w}_n(\psi)$$

$$+ (f_{11} + a_{11}\lambda)\, \dddot{w}_n(\psi) + (f_{12} + a_{12}\lambda)\, \ddddot{w}_n(\psi) = 0 \qquad (10b)$$

$$(g_1 + b_1\lambda)\, u_n(\psi) + (g_2 + b_2\lambda)\, \dot{u}_n(\psi) + (g_3 + b_3\lambda)\, \ddot{u}_n(\psi)$$

$$+ (g_4 + b_4\lambda)\, v_n(\psi) + (g_5 + b_5\lambda)\, \dot{v}_n(\psi) + (g_6 + b_6\lambda)\, \ddot{v}_n(\psi)$$

$$+ (g_7 + b_7\lambda)\, \dddot{v}_n(\psi) + (g_8 + b_8\lambda)\, w_n(\psi) + (g_9 + b_9\lambda)\, \dot{w}_n(\psi)$$

$$+ (g_{10} + b_{10}\lambda)\, \ddot{w}_n(\psi) + (g_{11} + b_{11}\lambda)\, \dddot{w}_n(\psi) + (g_{12} + b_{12}\lambda)\, \ddddot{w}_n(\psi) = 0$$

(10c)

In Eqs. (10) the coefficients a , b , c , f , g , and h , which are now different from those defined in Eqs. (V-9), are given by

$$h_1 = nk\Big[n^2C_1 - 4C_1S_1^2 - (\alpha + C_1)\, C_1^2 - (2 - \nu)(\alpha + C_1)\, C_2$$

$$+ (1 + \nu)(\alpha + C_1)\, S_1^2 + (\alpha + C_1)^2\, C_1\Big]$$

$$+ n\Big[(\alpha + C_1)^2\, C_1 + \nu\, (\alpha + C_1)^3\Big] \qquad (11)$$

$$h_2 = nk\Big[-3\,(\alpha + C_1)\, C_1S_1 + 2\,(\alpha + C_1)^2\, S_1\Big]$$

$$h_3 = -nk\,(\alpha + C_1)^2\, C_1$$

$$h_4 = k\Big[+(\alpha + C_1)\, S_1^3 - n^2\,(\alpha + C_1)\, S_1 + 2\,(\alpha + C_1)^2\, C_1S_1$$

$$- \nu(\alpha + C_1)^3\, S_1\Big] - (\alpha + C_1)^2\, C_1S_1 - \nu(\alpha + C_1)^3\, S_1$$

$$h_5 = k\Big[n^2\,(\alpha + C_1)^2 + (\alpha + C_1)^2\, S_1^2 + (1 + \nu)(\alpha + C_1)^3\, C_1\Big]$$

$$+ \nu\,(\alpha + C_1)^3\, C_1 + (\alpha + C_1)^4$$

$$h_6 = +2k\,(\alpha + C_1)^3\, S_1$$

$$h_7 = -k\,(\alpha + C_1)^4 \qquad \text{(11 cont'd)}$$

$$h_8 = n^2 k \left[n^2 - 4S_1^{\,2} - (3 - \nu)(\alpha + C_1)\,C_1 \right] + (\alpha + C_1)^2\,C_1^{\,2}$$

$$+ 2\nu(\alpha + C_1)^3\,C_1 + (\alpha + C_1)^4$$

$$h_9 = k \left[-2n^2 (\alpha + C_1)\,S_1 - (\alpha + C_1)\,S_1^{\,3} - 2(\alpha + C_1)^2\,C_1 S_1 \right.$$

$$\left. + \nu(\alpha + C_1)^3\,S_1 \right]$$

$$h_{10} = k \left[-2n^2 (\alpha + C_1)^2 - (\alpha + C_1)^2\,S_1^{\,2} - (1 + \nu)(\alpha + C_1)^3\,C_1 \right]$$

$$h_{11} = -2k\,(\alpha + C_1)^3\,S_1$$

$$h_{12} = +k\,(\alpha + C_1)^4$$

$$f_1 = +\frac{1}{2}(3 - \nu)\,nS_1$$

$$f_2 = +\frac{1}{2}(1 + \nu)\,n\,(\alpha + C_1)$$

$$f_3 = 0$$

$$f_4 = -\nu(\alpha + C_1)\,C_1 - \frac{1}{2}(1 - \nu)\,n^2 - S_1^{\,2}$$

$$f_5 = -(\alpha + C_1)\,S_1$$

$$f_6 = (\alpha + C_1)^2$$

$$f_7 = 0$$

$$f_8 = -\,\alpha S_1$$

$$f_9 = (\alpha + C_1)^2 + \nu(\alpha + C_1)\,C_1$$

$$f_{10} = f_{11} = f_{12} = 0$$

$$g_1 = -n^2(\alpha + C_1) - \frac{1}{2}(1-\nu)(\alpha + C_1)S_1{}^2 + \frac{1}{2}(1-\nu)(\alpha + C_1)^2 C_1 \qquad \text{(11 cont'd)}$$

$$g_2 = -\frac{1}{2}(1-\nu)(\alpha + C_1)^2 S_1$$

$$g_3 = +\frac{1}{2}(1-\nu)(\alpha + C_1)^3$$

$$g_4 = +\frac{1}{2}(3-\nu)\,n\,(\alpha + C_1)S_1$$

$$g_5 = -\frac{1}{2}(1+\nu)\,n\,(\alpha + C_1)^2$$

$$g_6 = g_7 = 0$$

$$g_8 = -n(\alpha + C_1)C_1 - \nu n(\alpha + C_1)^2$$

$$g_9 = g_{10} = g_{11} = g_{12} = 0$$

$$c_1 = (1-\nu^2)\,n\left[-\frac{1}{2}(\alpha + C_1)^2 C_1 + (\alpha + C_1)^3\right]$$

$$c_2 = c_3 = 0$$

$$c_4 = -\frac{1}{2}(1-\nu^2)(\alpha + C_1)^3 S_1$$

$$c_5 = (1-\nu^2)\left[-\frac{1}{2}(2\alpha + C_1)(\alpha + C_1)^3 + (\alpha + C_1)^4\right]$$

$$c_6 = c_7 = 0$$

$$c_8 = (1-\nu^2)\left[-\frac{1}{2}n^2(\alpha + C_1)^2 + (\alpha + C_1)^3 C_1 + (\alpha + C_1)^4\right]$$

$$c_9 = -\frac{1}{2}(1-\nu^2)(\alpha + C_1)^3 S_1$$

$$c_{10} = +\frac{1}{2}(1-\nu^2)(2\alpha + C_1)(\alpha + C_1)^3$$

$$c_{11} = c_{12} = 0$$

$$a_1 = a_2 = a_3 = 0$$

$$a_4 = (1-\nu^2)\left[\frac{1}{2}(2\alpha + C_1)(\alpha + C_1) - (\alpha + C_1)^2 + \frac{1}{2}n^2\right]$$

$$a_5 = a_6 = a_7 = a_8 = 0 \qquad \text{(11 concl'd)}$$

$$a_9 = (1 - \nu^2)\left[-\frac{1}{2}(2\alpha + C_1)(\alpha + C_1) + (\alpha + C_1)^2 \right]$$

$$a_{10} = a_{11} = a_{12} = 0$$

$$b_1 = (1-\nu^2)\left[\frac{1}{2}(\alpha + C_1)C_1^{\,2} - (\alpha + C_1)^2 C_1 - \frac{1}{2}(2\alpha + C_1)(\alpha + C_1)C_1 - \frac{1}{2}\alpha S_1^{\,2}\right]$$

$$b_2 = \frac{1}{2}(1 - \nu^2)\left[(\alpha + C_1)^2 S_1 - (2\alpha + C_1)(\alpha + C_1) S_1\right]$$

$$b_3 = -\frac{1}{2}(1 - \nu^2)(2\alpha + C_1)(\alpha + C_1)^2$$

$$b_4 = -\frac{1}{2}(1 - \nu^2)\, n\, (\alpha + C_1) S_1$$

$$b_5 = b_6 = b_7 = 0$$

$$b_8 = (1 - \nu^2)\left[+\frac{1}{2} n (\alpha + C_1) C_1 - n(\alpha + C_1)^2\right]$$

$$b_9 = b_{10} = b_{11} = b_{12} = 0$$

2. Solution of the Stability Equations

2.1 Outline of Method of Solution

The stability equations for a toroidal shell [Eqs. (10)] consist of three linear homogeneous ordinary differential equations with variable coefficients. The unknowns in these equations are the three displacement components $u_n(\psi)$, $v_n(\psi)$, and $w_n(\psi)$. For a complete toroidal shell, the boundary conditions are simply conditions of periodicity on the displacement components; hence, $u_n(\psi)$, $v_n(\psi)$, and $w_n(\psi)$ may be represented by Fourier series in the meridional coordinate ψ. Next, in order to pave the way for a Fourier series analysis, the coefficients in the stability equations are expressed as linear combinations of trigonometric functions. Then, with the aid of some identities which will be

derived, it is possible to write each of the three stability equations in the following form:

$$\sum_{m=0}^{\infty} \Big\{ \Big\}_m \cos(m\psi) + \sum_{m=1}^{\infty} \Big\{ \Big\}_m \sin(m\psi) = 0 \quad . \tag{12}$$

The braced expressions in Eqs. (12) represent homogeneous linear combinations of the Fourier coefficients used in the expansions for u_n , v_n , and w_n . The trigonometric functions in Eqs. (12) are linearly independent for $0 \le \psi \le 2\pi$; hence, each of the braced expressions must vanish. Thus the problem of solving a system of ordinary differential equations with variable coefficients is reduced to that of solving an infinite system of linear homogeneous algebraic equations. A matrix iteration technique is used to get the lowest eigenvalue of a finite system of equations which is obtained through truncation of the infinite system of equations. The size of the finite system of equations is then successively increased until no significant change occurs in the computed eigenvalue.

2.2 The Solution

In order to facilitate the subsequent Fourier series analysis, we now express the coefficients in the stability equations as linear combinations of trigonometric functions. This is effected by employing the well known trigonometric identities

$$C_a S_b = \frac{1}{2}\left[S_{a+b} - S_{a-b}\right]$$

$$C_a C_b = \frac{1}{2}\left[C_{a+b} + C_{a-b}\right]$$

$$S_a S_b = \frac{1}{2}\left[C_{a-b} - C_{a+b}\right]$$

from which it follows that

$$(\alpha + C_1) C_1 = \frac{1}{2} + \alpha C_1 + \frac{1}{2} C_2 \tag{13a}$$

$$(\alpha + C_1) C_2 = \frac{1}{2} C_1 + \alpha C_2 + \frac{1}{2} C_3 \tag{13b}$$

$$(\alpha + C_1) S_1^2 = \frac{\alpha}{2} + \frac{1}{4} C_1 - \frac{\alpha}{2} C_2 - \frac{1}{4} C_3 \tag{13c}$$

$$(\alpha + C_1) S_1^3 = \frac{3}{4}\alpha S_1 + \frac{1}{4} S_2 - \frac{1}{4}\alpha S_3 - \frac{1}{8} S_4 \tag{13d}$$

$$(\alpha + C_1)^2 C_1 = \alpha + \left(\frac{3}{4} + \alpha^2\right) C_1 + \alpha C_2 + \frac{1}{4} C_3 \tag{13e}$$

$$(\alpha + C_1)^2 C_1^2 = \frac{3}{8} + \frac{\alpha^2}{2} + \frac{3}{2}\alpha C_1 + \left(\frac{1}{2} + \frac{\alpha^2}{2}\right) C_2 + \frac{\alpha}{2} C_3 + \frac{1}{8} C_4 \tag{13f}$$

$$(\alpha + C_1)^2 S_1^2 = \frac{1}{8} + \frac{\alpha^2}{2} + \frac{\alpha}{2} C_1 - \frac{\alpha^2}{2} C_2 - \frac{\alpha}{2} C_3 - \frac{1}{8} C_4 \tag{13g}$$

$$(\alpha + C_1)^2 C_1 S_1 = \frac{\alpha}{2} S_1 + \left(\frac{1}{4} + \frac{\alpha^2}{2}\right) S_2 + \frac{\alpha}{2} S_3 + \frac{1}{8} S_4 \tag{13h}$$

$$(\alpha + C_1)^3 = \frac{3}{2}\alpha + \alpha^3 + \left(\frac{3}{4} + 3\alpha^2\right) C_1 + \frac{3}{2}\alpha C_2 + \frac{1}{4} C_3 \tag{13i}$$

$$(\alpha + C_1)^3 C_1 = \frac{3}{8} + \frac{3}{2}\alpha^2 + \left(\frac{9}{4}\alpha + \alpha^3\right) C_1 + \left(\frac{1}{2} + \frac{3}{2}\alpha^2\right) C_2$$

$$+ \frac{3}{4}\alpha C_3 + \frac{1}{8} C_4 \tag{13j}$$

$$(\alpha + C_1)^3 S_1 = \left(\frac{3}{4}\alpha + \alpha^3\right) S_1 + \left(\frac{1}{4} + \frac{3}{2}\alpha^2\right) S_2 + \frac{3}{4}\alpha S_3 + \frac{1}{8} S_4 \tag{13k}$$

$$(\alpha + C_1)^4 = \frac{3}{8} + 3\alpha^2 + \alpha^4 + (3\alpha + 4\alpha^3) C_1 + \left(\frac{1}{2} + 3\alpha^2\right) C_2$$

$$+ \alpha C_3 + \frac{1}{8} C_4 \tag{13l}$$

$$(\alpha + C_1) S_1 = \alpha S_1 + \frac{1}{2} S_2 \tag{13m}$$

$$(\alpha + C_1) C_1^2 = \frac{\alpha}{2} + \frac{3}{4} C_1 + \frac{\alpha}{2} C_2 + \frac{1}{4} C_3 \tag{13n}$$

$$(\alpha + C_1) C_1 S_1 = \frac{1}{4} S_1 + \frac{\alpha}{2} S_2 + \frac{1}{4} S_3 \tag{13o}$$

$$(\alpha + C_1)^2 = \frac{1}{2} + \alpha^2 + 2\alpha C_1 + \frac{1}{2} C_2 \tag{13p}$$

$$(\alpha + C_1)^2 S_1 = \left(\frac{1}{4} + \alpha^2\right) S_1 + \alpha S_2 + \frac{1}{4} S_3 \tag{13q}$$

Insertion of Eqs. (13) into Eqs. (10) and (11) yields the following form of the stability equations:

$$\begin{aligned}
&\Big[(h_{10} + c_{10}\lambda) + (h_{11} + c_{11}\lambda)\, C_1 + (h_{12} + c_{12}\lambda)\, C_2 + (h_{13} + c_{13}\lambda)\, C_3\Big]\, u_n(\psi) \\
&\quad - \Big[h_{21}\, S_1 + h_{22}\, S_2 + h_{23}\, S_3\Big]\, \dot{u}_n(\psi) - \Big[(h_{30} + h_{31}\, C_1 + h_{32}\, C_2 \\
&\quad + h_{33}\, C_3\Big]\, \ddot{u}_n(\psi) + \Big[(h_{41} + c_{41}\lambda)\, S_1 + (h_{42} + c_{42}\lambda)\, S_2 + (h_{43} + c_{43}\lambda)\, S_3 \\
&\quad + (h_{44} + c_{44}\lambda)\, S_4\Big]\, v_n(\psi) + \Big[(h_{50} + c_{50}\lambda) + (h_{51} + c_{51}\lambda)\, C_1 \\
&\quad + (h_{52} + c_{52}\lambda)\, C_2 + (h_{53} + c_{53}\lambda)\, C_3 + (h_{54} + c_{54}\lambda)\, C_4\Big]\, \dot{v}_n(\psi) \\
&\quad - \Big[h_{61}\, S_1 + h_{62}\, S_2 + h_{63}\, S_3 + h_{64}\, S_4\Big]\, \ddot{v}_n(\psi) - \Big[h_{70} + h_{71}\, C_1 + h_{72}\, C_2 \\
&\quad + h_{73}\, C_3 + h_{74}\, C_4\Big]\, \bar{v}_n(\psi) + \Big[(h_{80} + c_{80}\lambda) + (h_{81} + c_{81}\lambda)\, C_1 \\
&\quad + (h_{82} + c_{82}\lambda)\, C_2 + (h_{83} + c_{83}\lambda)\, C_3 + (h_{84} + c_{84}\lambda)\, C_4\Big]\, w_n(\psi) \\
&\quad - \Big[(h_{91} + c_{91}\lambda)\, S_1 + (h_{92} + c_{92}\lambda)\, S_2 + (h_{93} + c_{93}\lambda)\, S_3 \\
&\quad + (h_{94} + c_{94}\lambda)\, S_4\Big]\, \dot{w}_n(\psi) - \Big[(h_{10,0} + c_{10,0}\lambda) + h_{10,1} + c_{10,1}\lambda)\, C_1 \\
&\quad + (h_{10,2} + c_{10,2}\lambda)\, C_2 + (h_{10,3} + c_{10,3}\lambda)\, C_3 + (h_{10,4} + c_{10,4}\lambda)\, C_4\Big]\, \ddot{w}_n(\psi) \\
&\quad + \Big[h_{11,1}\, S_1 + h_{11,2}\, S_2 + h_{11,3}\, S_3 + h_{11,4}\, S_4\Big]\, \dddot{w}_n(\psi) + \Big[h_{12,0} \\
&\quad + h_{12,1}\, C_1 + h_{12,2}\, C_2 + h_{12,3}\, C_3 + h_{12,4}\, C_4\Big]\, \bar{w}_n(\psi) = 0
\end{aligned} \tag{14a}$$

$$\begin{aligned}
&\Big[f_{11}\, S_1\Big]\, u_n(\psi) - \Big[f_{20} + f_{21}\, C_1\Big]\, \dot{u}_n(\psi) + \Big[(f_{40} + a_{40}\lambda) + (f_{41} + a_{41}\lambda)\, C_1 \\
&\quad + (f_{42} + a_{42}\lambda)\, C_2\Big]\, v_n(\psi) + \Big[f_{51}\, S_1 + f_{52}\, S_2\Big]\, \dot{v}_n(\psi) \\
&\quad - \Big[f_{60} + f_{61}\, C_1 + f_{62}\, C_2\Big]\, \ddot{v}_n(\psi) + \Big[f_{81}\, S_1\Big]\, w_n(\psi) - \Big[(f_{90} + a_{90}\lambda) \\
&\quad + (f_{91} + a_{91}\lambda)\, C_1 + (f_{92} + a_{92}\lambda)\, C_2\Big]\, \dot{w}_n(\psi) = 0
\end{aligned} \tag{14b}$$

$$\Big[(g_{10} + b_{10}\lambda) + (g_{11} + b_{11}\lambda) C_1 + (g_{12} + b_{12}\lambda) C_2 + (g_{13} + b_{13}\lambda) C_3\Big] u_n(\psi)$$

$$- \Big[(g_{21} + b_{21}\lambda) S_1 + (g_{22} + b_{22}\lambda) S_2 + (g_{23} + b_{23}\lambda) S_3\Big] \dot{u}_n(\psi)$$

$$- \Big[(g_{30} + b_{30}\lambda) + (g_{31} + b_{31}\lambda) C_1 + (g_{32} + b_{32}\lambda) C_2$$

$$+ (g_{33} + b_{33}\lambda) C_3\Big] \ddot{u}_n(\psi) + \Big[(g_{41} + b_{41}\lambda) S_1 + (g_{42} + b_{42}\lambda) S_2\Big] v_n(\psi)$$

$$+ \Big[g_{50} + g_{51} C_1 + g_{52} C_2\Big] \dot{v}_n(\psi) + \Big[(g_{80} + b_{80}\lambda) + (g_{81} + b_{81}\lambda) C_1$$

$$+ (g_{82} + b_{82}\lambda) C_2\Big] w_n(\psi) \quad = \quad 0 \tag{14c}$$

where

$$h_{10} = \left[\left(\frac{1 + \nu}{2}\right) k + \left(1 + \frac{3}{2}\nu\right) + \nu\alpha^2\right] \alpha n$$

$$h_{11} = \left\{\left[\left(-\frac{7}{4} + \frac{3}{4}\nu\right) + \alpha^2 + n^2\right] k + \frac{3}{4}(1 + \nu) + (1 + 3\nu)\alpha^2\right\} n$$

$$h_{12} = \left[\left(-2 + \frac{\nu}{2}\right) k + \left(1 + \frac{3}{2}\nu\right)\right] \alpha n$$

$$h_{13} = \left[-\frac{1}{4}(1 - \nu) k + \frac{1}{4}(1 + \nu)\right] n$$

$$h_{21} = \left(-\frac{1}{4} + 2\alpha^2\right) nk$$

$$h_{22} = +\frac{1}{2}\alpha nk$$

$$h_{23} = -\frac{1}{4} nk$$

$$h_{30} = -\alpha nk$$

$$h_{31} = -\left(\frac{3}{4} + \alpha^2\right) nk$$

$$h_{32} = -\alpha nk$$

$$h_{33} = -\frac{1}{4} nk$$

$$h_{41} = \left\{\left[\left(\frac{7}{4} - \frac{3}{4}\nu\right) - n^2 - \nu\alpha^2\right]k - \frac{1}{2} - \frac{3}{4}\nu - \nu\alpha^2\right\}\alpha \qquad \text{(15 cont'd)}$$

$$h_{42} = \left[\frac{1}{4}(3 - \nu) - \frac{1}{2}n^2 + \left(1 - \frac{3}{2}\nu\right)\alpha^2\right]k - \frac{1}{4}(1 + \nu) - \frac{1}{2}(1 + 3\nu)\alpha^2$$

$$h_{43} = \left[\frac{3}{4}(1 - \nu)k - \left(\frac{1}{2} + \frac{3}{4}\nu\right)\right]\alpha$$

$$h_{44} = \frac{1}{8}(1 - \nu)k - \frac{1}{8}(1 + \nu)$$

$$h_{50} = \left[\left(\frac{1}{2} + \frac{3}{8}\nu\right) + \left(2 + \frac{3}{2}\nu\right)\alpha^2 + \left(\frac{1}{2} + \alpha^2\right)n^2\right]k + \frac{3}{8}(1 + \nu)$$
$$+ \left(3 + \frac{3}{2}\nu + \alpha^2\right)\alpha^2$$

$$h_{51} = \left\{\left[\frac{1}{4}(11 + 9\nu) + (1 + \nu)\alpha^2 + 2n^2\right]k + \left(3 + \frac{9}{4}\nu\right)\right.$$
$$\left. + (4 + \nu)\alpha^2\right\}\alpha$$

$$h_{52} = \left[\frac{1}{2}(1 + \nu) + \left(1 + \frac{3}{2}\nu\right)\alpha^2 + \frac{1}{2}n^2\right]k + \frac{1}{2}(1 + \nu) + \left(3 + \frac{3}{2}\nu\right)\alpha^2$$

$$h_{53} = \left[\left(\frac{1}{4} + \frac{3}{4}\nu\right)k + \left(1 + \frac{3}{4}\nu\right)\right]\alpha$$

$$h_{54} = \frac{1}{8}\nu k + \frac{1}{8}(1 + \nu)$$

$$h_{61} = +\left(\frac{3}{2} + 2\alpha^2\right)\alpha k$$

$$h_{62} = +\left(\frac{1}{2} + 3\alpha^2\right)k$$

$$h_{63} = +\frac{3}{2}\alpha k$$

$$h_{64} = +\frac{1}{4}k$$

$$h_{70} = -\left(\frac{3}{8} + 3\alpha^2 + \alpha^4\right)k$$

$$h_{71} = -(3 + 4\alpha^2)\alpha k$$

$$h_{72} = -\left(\frac{1}{2} + 3\alpha^2\right)k$$

$$h_{73} = -\alpha k \qquad (15\text{ cont'd})$$

$$h_{74} = -\frac{1}{8}k$$

$$h_{80} = \left[\frac{1}{2}(-7 + \nu) + n^2\right]n^2k + \frac{3}{4}(1 + \nu) + \left(\frac{7}{2} + 3\nu + \alpha^2\right)\alpha^2$$

$$h_{81} = \left[-(3 - \nu)\,n^2k + \frac{9}{2}(1 + \nu) + (4 + 2\nu)\,\alpha^2\right]\alpha$$

$$h_{82} = \frac{1}{2}(1 + \nu)\,n^2k + (1 + \nu) + \left(\frac{7}{2} + 3\,\nu\right)\alpha^2$$

$$h_{83} = \frac{3}{2}(1 + \nu)\,\alpha$$

$$h_{84} = \frac{1}{4}(1 + \nu)$$

$$h_{91} = \left[\frac{1}{4}(-7 + 3\nu) + \nu\alpha^2 - 2n^2\right]\alpha k$$

$$h_{92} = \left[\frac{1}{4}(-3 + \nu) + \left(-1 + \frac{3}{2}\nu\right)\alpha^2 - n^2\right]k$$

$$h_{93} = -\frac{3}{4}(1 - \nu)\,\alpha k$$

$$h_{94} = -\frac{1}{8}(1 - \nu)\,k$$

$$h_{10,0} = \left[\left(-\frac{1}{2} - \frac{3}{8}\nu\right) - \left(2 + \frac{3}{2}\nu + 2n^2\right)\alpha^2 - n^2\right]k$$

$$h_{10,1} = \left[-\frac{1}{4}(11 + 9\nu) - (1 + \nu)\,\alpha^2 - 4n^2\right]\alpha k$$

$$h_{10,2} = \left[-\frac{1}{2}(1 + \nu) - \left(1 + \frac{3}{2}\nu\right)\alpha^2 - n^2\right]k$$

$$h_{10,3} = -\frac{1}{4}(1 + 3\nu)\,\alpha k$$

$$h_{10,4} = -\frac{1}{8}\nu k$$

$$h_{11,1} = \left(-\frac{3}{2} + 2\alpha^2\right)\alpha k$$

$$h_{11,2} = -\left(\frac{1}{2} + 3\alpha^2\right)k$$

$$h_{11,3} = -\frac{3}{2}\alpha k \tag{15 cont'd}$$

$$h_{11,4} = -\frac{1}{4}k$$

$$h_{12,0} = \left(\frac{3}{8} + 3\alpha^2 + \alpha^4\right)k$$

$$h_{12,1} = (3 + 4\alpha^2)\,\alpha k$$

$$h_{12,2} = \left(\frac{1}{2} + 3\alpha^2\right)k$$

$$h_{12,3} = \alpha k$$

$$h_{12,4} = \frac{1}{8}k$$

$$c_{10} = +(1 - \nu^2)\,(1 + \alpha^2)\,\alpha n$$

$$c_{11} = +(1 - \nu^2)\left(\frac{3}{8} + \frac{5}{2}\alpha^2\right)n$$

$$c_{12} = +(1 - \nu^2)\,\alpha n$$

$$c_{13} = +\frac{1}{8}(1 - \nu^2)\,n$$

$$c_{41} = -\frac{1}{2}(1 - \nu^2)\left(\frac{3}{4} + \alpha^2\right)\alpha$$

$$c_{42} = -\frac{1}{2}(1 - \nu^2)\left(\frac{1}{4} + \frac{3}{2}\alpha^2\right)$$

$$c_{43} = -\frac{3}{8}(1 - \nu^2)\,\alpha$$

$$c_{44} = -\frac{1}{16}(1 - \nu^2)$$

$$c_{50} = (1 - \nu^2)\left(\frac{3}{16} + \frac{3}{4}\alpha^2\right)$$

$$c_{51} = (1 - \nu^2)\left(\frac{9}{8} + \frac{1}{2}\alpha^2\right)\alpha$$

$$c_{52} = (1 - \nu^2)\left(\frac{1}{4} + \frac{3}{4}\alpha^2\right)$$

$$c_{53} = \frac{3}{8}(1 - \nu^2)\alpha$$ (15 cont'd)

$$c_{54} = +\frac{1}{16}(1 - \nu^2)$$

$$c_{80} = (1 - \nu^2)\left[\frac{3}{4} + \left(\frac{9}{2} + \alpha^2 - \frac{1}{2}n^2\right)\alpha^2 - \frac{1}{4}n^2\right]$$

$$c_{81} = (1 - \nu^2)\left(\frac{21}{4} + 5\alpha^2 - n^2\right)\alpha$$

$$c_{82} = (1 - \nu^2)\left(1 + \frac{9}{2}\alpha^2 - \frac{1}{4}n^2\right)$$

$$c_{83} = +\frac{7}{4}(1 - \nu^2)\alpha$$

$$c_{84} = +\frac{1}{4}(1 - \nu^2)$$

$$c_{91} = -(1 - \nu^2)\left(\frac{3}{8} + \frac{1}{2}\alpha^2\right)\alpha$$

$$c_{92} = -(1 - \nu^2)\left(\frac{1}{8} + \frac{3}{4}\alpha^2\right)$$

$$c_{93} = -\frac{3}{8}(1 - \nu^2)\alpha$$

$$c_{94} = -\frac{1}{16}(1 - \nu^2)$$

$$c_{10,0} = (1 - \nu^2)\left(\frac{3}{16} + \frac{9}{4}\alpha^2 + \alpha^4\right)$$

$$c_{10,1} = (1 - \nu^2)\left(\frac{15}{8} + \frac{7}{2}\alpha^2\right)\alpha$$

$$c_{10,2} = (1 - \nu^2)\left(\frac{1}{4} + \frac{9}{4}\alpha^2\right)$$

$$c_{10,3} = +\frac{5}{8}(1 - \nu^2)\alpha$$

$$c_{10,4} = +\frac{1}{16}(1 - \nu^2)$$

$$f_{11} = +\frac{1}{2}(3 - \nu)\,n$$

$$f_{20} = +\frac{1}{2}(1 + \nu)\,n\alpha$$

$$f_{21} = +\frac{1}{2}(1 + \nu)\, n \qquad \text{(15 cont'd)}$$

$$f_{40} = -\frac{1}{2}\left[(1 + \nu) + (1 - \nu)\, n^2\right]$$

$$f_{41} = -\nu\alpha$$

$$f_{42} = +\frac{1}{2}(1 - \nu)$$

$$f_{51} = -\alpha$$

$$f_{52} = -\frac{1}{2}$$

$$f_{60} = +\left(\frac{1}{2} + \alpha^2\right)$$

$$f_{61} = +2\alpha$$

$$f_{62} = +\frac{1}{2}$$

$$f_{81} = -\alpha$$

$$f_{90} = +\frac{1}{2}(1 + \nu) + \alpha^2$$

$$f_{91} = +(2 + \nu)\alpha$$

$$f_{92} = +\frac{1}{2}(1 + \nu)$$

$$a_{40} = (1 - \nu^2)\left(-\frac{1}{4} + \frac{1}{2} n^2\right)$$

$$a_{41} = -\frac{1}{2}(1 - \nu^2)\alpha$$

$$a_{42} = -\frac{1}{4}(1 - \nu^2)$$

$$a_{90} = \frac{1}{4}(1 - \nu^2)$$

$$a_{91} = +\frac{1}{2}(1 - \nu^2)\,\alpha \qquad \text{(15 cont'd)}$$

$$a_{92} = +\frac{1}{4}(1 - \nu^2)$$

$$g_{10} = \left[\frac{1}{4}(1 - \nu) - n^2\right]\alpha$$

$$g_{11} = (1 - \nu)\left(\frac{1}{4} + \frac{\alpha^2}{2}\right) - n^2$$

$$g_{12} = \frac{3}{4}(1 - \nu)\,\alpha$$

$$g_{13} = \frac{1}{4}(1 - \nu)$$

$$g_{30} = (1 - \nu)\left(\frac{3}{4} + \frac{\alpha^2}{2}\right)\alpha$$

$$g_{31} = (1 - \nu)\left(\frac{3}{8} + \frac{3}{2}\alpha^2\right)$$

$$g_{32} = \frac{3}{4}(1 - \nu)\alpha$$

$$g_{33} = \frac{1}{8}(1 - \nu)$$

$$g_{41} = \frac{1}{2}(3 - \nu)\,n\alpha$$

$$g_{42} = \frac{1}{4}(3 - \nu)\,n$$

$$g_{80} = -\left[\frac{1}{2}(1 + \nu) + \nu\alpha^2\right]n$$

$$g_{81} = -(1 + 2\nu)\,\alpha n$$

$$g_{82} = -\frac{1}{2}(1 + \nu)\,n$$

$$g_{21} = -(1 - \nu)\left(\frac{1}{8} + \frac{\alpha^2}{2}\right)$$

$$g_{22} = -(1 - \nu)\frac{\alpha}{2}$$

$$g_{23} = -\frac{1}{8}(1 - \nu)$$

$$g_{50} = -(1 + \nu)\, n\left(\frac{1}{4} + \frac{\alpha^2}{2}\right) \qquad \text{(15 cont'd)}$$

$$g_{51} = -(1 + \nu)\, n\alpha$$

$$g_{52} = -\frac{1}{4}(1 + \nu)\, n$$

$$b_{10} = -\frac{7}{4}(1 - \nu^2)\,\alpha$$

$$b_{11} = -(1 - \nu^2)\left(\frac{3}{4} + 2\alpha^2\right)$$

$$b_{12} = -\frac{5}{4}(1 - \nu^2)\,\alpha$$

$$b_{13} = -\frac{1}{4}(1 - \nu^2)$$

$$b_{21} = -\frac{1}{2}(1 - \nu^2)\,\alpha^2$$

$$b_{22} = -\frac{1}{4}(1 - \nu^2)\,\alpha$$

$$b_{23} = 0$$

$$b_{41} = -\frac{1}{2}(1 - \nu^2)\, n\alpha$$

$$b_{42} = -\frac{1}{4}(1 - \nu^2)\, n$$

$$b_{80} = -(1 - \nu^2)\left(\frac{1}{4} + \alpha^2\right) n$$

$$b_{81} = -\frac{3}{2}(1 - \nu^2)\,\alpha n$$

$$b_{82} = -\frac{1}{4}(1 - \nu^2)\, n$$

$$b_{30} = -(1 - \nu^2)\,(1 + \alpha^2)\,\alpha$$

$$b_{31} = -(1 - \nu^2)\left(\frac{3}{8} + \frac{5}{2}\alpha^2\right)$$

$$b_{32} = -(1 - \nu^2)\alpha \qquad \text{(15 concl'd)}$$

$$b_{33} = -\frac{1}{8}(1 - \nu^2)$$

For each value of n , the displacement components $u_n(\psi)$, $v_n(\psi)$, and $w_n(\psi)$ may be represented by the Fourier series:

$$u_n(\psi) = \sum_{m=0}^{\infty} U_m \cos m\psi + \sum_{m=1}^{\infty} \tilde{U}_m \sin m\psi \qquad \text{(16a)}$$

$$v_n(\psi) = \sum_{m=1}^{\infty} V_m \sin m\psi + \sum_{m=0}^{\infty} \tilde{V}_m \cos m\psi \qquad \text{(16b)}$$

$$w_n(\psi) = \sum_{m=0}^{\infty} W_m \cos m\psi + \sum_{m=1}^{\infty} \tilde{W}_m \sin m\psi \quad . \qquad \text{(16c)}$$

The series with the Fourier coefficients U_m , V_m , W_m represent a buckling mode which is symmetric about the plane $\psi = 0, \pi$ (plane A-A in Fig. 1), whereas the series with the Fourier coefficients $\tilde{U}_m$, $\tilde{V}_m$, $\tilde{W}_m$ represent a buckling mode which is antimetric about the plane $\psi = 0, \pi$. After insertion of Eqs. (16) into Eqs. (14), it can be shown that the resulting stability equations may be put in the form:

$$\sum_{m=0}^{\infty} A_m \cos m\psi + \sum_{m=1}^{\infty} \tilde{A}_m \sin m\psi = 0 \qquad \text{(17a)}$$

$$\sum_{m=1}^{\infty} B_m \sin m\psi + \sum_{m=0}^{\infty} \tilde{B}_m \cos m\psi = 0 \qquad \text{(17b)}$$

$$\sum_{m=0}^{\infty} C_m \cos m\psi + \sum_{m=1}^{\infty} \tilde{C}_m \sin m\psi = 0 \qquad \text{(17c)}$$

where

$$A_m = A_m(U_m, V_m, W_m) \,, \quad (18a) \qquad \tilde{A}_m = \tilde{A}_m(\tilde{U}_m, \tilde{V}_m, \tilde{W}_m) \quad (18d)$$

$$B_m = B_m(U_m, V_m, W_m) \,, \quad (18b) \qquad \tilde{B}_m = \tilde{B}_m(\tilde{U}_m, \tilde{V}_m, \tilde{W}_m) \quad (18e)$$

$$C_m = C_m(U_m, V_m, W_m) \,, \quad (18d) \qquad \tilde{C}_m = \tilde{C}_m(\tilde{U}_m, \tilde{V}_m, \tilde{W}_m) \,. \quad (18f)$$

Since the functions $\sin(m\psi)$ and $\cos(m\psi)$ in Eqs. (17) are linearly independent for $0 \leq \psi \leq 2\pi$, we conclude that

$$A_m = B_m = C_m = 0 \tag{19a}$$

$$\tilde{A}_m = \tilde{B}_m = \tilde{C}_m = 0 \,. \tag{19b}$$

From Eqs. (18 and 19) we see that the Fourier coefficients U_m, V_m, W_m may be determined from a set of equations which do not contain $\tilde{U}_m$, $\tilde{V}_m$, $\tilde{W}_m$. This means that a toroidal shell under uniform external pressure can buckle into a mode which is either symmetric or antimetric about the plane $\psi = 0, \pi$, and thus these modes can be investigated separately. For convenience, the buckling mode which is symmetric about the plane $\psi = 0, \pi$ is called Mode A and the buckling mode which is antimetric about the plane $\psi = 0, \pi$ is called Mode B. These two modes are considered in the next two subsections.

2.2.1 Mode A

For the buckling mode which is symmetric about the plane $\psi = 0, \pi$, we let

$$u_n(\psi) = \sum_{m=0}^{\infty} U_m \cos m\psi = \sum_{m=0}^{\infty} U_m C_m \tag{20a}$$

$$v_n(\psi) = \sum_{m=1}^{\infty} V_m \sin m\psi = \sum_{m=1}^{\infty} V_m S_m \tag{20b}$$

$$w_n(\psi) = \sum_{m=0}^{\infty} W_m \cos m\psi = \sum_{m=0}^{\infty} W_m C_m \tag{20c}$$

where, for brevity, we have used the notations given by Eqs. (3). Substitution of the Fourier series expansions for the displacement components [Eqs. (20)] into the stability equations for a toroidal shell [Eqs. (14)] yields

$$\Big[(h_{10} + c_{10}\lambda) + (h_{11} + c_{11}\lambda)\, C_1 + (h_{12} + c_{12}\lambda)\, C_2$$

$$+ (h_{13} + c_{13}\lambda) C_3\Big] \sum_{m=0}^{\infty} U_m C_m - \Big[h_{21} S_1 + h_{22} S_2$$

$$+ h_{23} S_3\Big] \sum_{m=1}^{\infty} m\, U_m S_m - \Big[h_{30} + h_{31} C_1 + h_{32} C_2$$

$$+ h_{33} C_3\Big] \sum_{m=0}^{\infty} m^2 U_m C_m + \Big[(h_{41} + c_{41}\lambda)\, S_1 + (h_{42} + c_{42}\lambda)\, S_2$$

$$+ (h_{43} + c_{43}\lambda)\, S_3 + (h_{44} + c_{44}\lambda)\, S_4\Big] \sum_{m=1}^{\infty} V_m S_m$$

$$+ \Big[(h_{50} + c_{50}\lambda) + (h_{51} + c_{51}\lambda)\, C_1 + (h_{52} + c_{52}\lambda)\, C_2$$

$$+ (h_{53} + c_{53}\lambda)\, C_3 + (h_{54} + c_{54}\lambda)\, C_4\Big] \sum_{m=0}^{\infty} m\, V_m C_m$$

(cont'd)

$$- \left[h_{61} S_1 + h_{62} S_2 + h_{63} S_3 + h_{64} S_4 \right] \sum_{m=1}^{\infty} m^2 V_m S_m$$

$$- \left[h_{70} + h_{71} C_1 + h_{72} C_2 + h_{73} C_3 + h_{74} C_4 \right] \sum_{m=0}^{\infty} m^3 V_m C_m$$

$$+ \left[(h_{80} + c_{80}\lambda) + (h_{81} + c_{81}\lambda) C_1 + (h_{82} + c_{82}\lambda) C_2 + (h_{83} + c_{83}\lambda) C_3 \right.$$

$$\left. + (h_{84} + c_{84}\lambda) C_4 \right] \sum_{m=0}^{\infty} W_m C_m - \left[(h_{91} + c_{91}\lambda) S_1 + (h_{92} + c_{92}\lambda) S_2 \right.$$

$$\left. + (h_{93} + c_{93}\lambda) S_3 + (h_{94} + c_{94}\lambda) S_4 \right] \sum_{m=1}^{\infty} m W_m S_m$$

$$- \left[(h_{10,0} + c_{10,0}\lambda) + (h_{10,1} + c_{10,1}\lambda) C_1 + (h_{10,2} + c_{10,2}\lambda) C_2 \right.$$

$$\left. + (h_{10,3} + c_{10,3}\lambda) C_3 + (h_{10,4} + c_{10,4}\lambda) C_4 \right] \sum_{m=0}^{\infty} m^2 W_m C_m$$

$$+ \left[h_{11,1} S_1 + h_{11,2} S_2 + h_{11,3} S_3 + h_{11,4} S_4 \right] \sum_{m=1}^{\infty} m^3 W_m S_m$$

$$+ \left[h_{12,0} + h_{12,1} C_1 + h_{12,2} C_2 + h_{12,3} C_3 \right.$$

$$\left. + h_{12,4} C_4 \right] \sum_{m=0}^{\infty} m^4 W_m C_m = 0 \tag{21a}$$

$$f_{11}\, S_1 \sum_{m=0}^{\infty} U_m\, C_m - \left[f_{20} + f_{21}\, C_1\right] \sum_{m=1}^{\infty} m\, U_m\, S_m + \left[(f_{40} + a_{40}\lambda)\right.$$

$$+ (f_{41} + a_{4i}\lambda)\, C_1 + (f_{42} + a_{42}\lambda)\, C_2\Big] \sum_{m=1}^{\infty} V_m\, S_m$$

$$+ \left[f_{51}\, S_1 + f_{52}\, S_2\right] \sum_{m=0}^{\infty} m\, V_m\, C_m$$

$$- \left[f_{60} + f_{61}\, C_1 + f_{62}\, C_2\right] \sum_{m=1}^{\infty} m^2\, V_m\, S_m$$

$$+ f_{81}\, S_1 \sum_{m=0}^{\infty} W_m\, C_m - \Big[(f_{90} + a_{90}\lambda) + (f_{91} + a_{91}\lambda)\, C_1$$

$$+ (f_{92} + a_{92}\lambda)\, C_2\Big] \sum_{m=1}^{\infty} m\, W_m\, S_m = 0 \qquad \text{(21b)}$$

$$\Big[(g_{10} + b_{10}\lambda) + (g_{11} + b_{11}\lambda)\, C_1 + (g_{12} + b_{12}\lambda)\, C_2 + (g_{13} + b_{13}\lambda)\, C_3\Big] \sum_{m=0}^{\infty} U_m\, C_m$$

$$- \Big[(g_{21} + b_{21}\lambda)\, S_1 + (g_{22} + b_{22}\lambda)\, S_2 + g_{23}\, S_3\Big] \sum_{m=1}^{\infty} m\, U_m\, S_m$$

$$- \Big[(g_{30} + b_{30}\lambda) + (g_{31} + b_{31}\lambda)\, C_1 + (g_{32} + b_{32}\lambda)\, C_2$$

$$+ (g_{33} + b_{33}\lambda)\, C_3\Big] \sum_{m=0}^{\infty} m^2\, U_m\, C_m + \Big[(g_{41} + b_{41}\lambda)\, S_1$$

$$+ (g_{42} + b_{42}\lambda)\, S_2\Big] \sum_{m=0}^{\infty} V_m\, S_m + \left[g_{50} + g_{51}\, C_1 + g_{52}\, C_2\right] \sum_{m=0}^{\infty} m\, V_m\, C_m$$

$$+ \Big[(g_{80} + b_{80}\lambda) + (g_{81} + b_{81}\lambda)\, C_1 + (g_{82} + b_{82}\lambda)\, C_2\Big] \sum_{m=0}^{\infty} W_m\, C_m = 0$$

(21c)

The terms in Eqs. (21) consist of products of trigonometric functions and Fourier series. These terms can be put into the form

$$\sum_{m=0}^{\infty} [\quad]_m C_m \text{ , or, } \sum_{m=1}^{\infty} [\quad]_m S_m$$

by use of some transformation equations which we will now derive. As an example, let us consider a term of the form

$$P = S_r \sum_{m=1}^{\infty} m^3 W_m S_m \text{ , } (1 \leq r \leq 4) \quad . \tag{22}$$

We use the trigonometric identity

$$2S_a S_b = C_{a-b} - C_{a+b}$$

to write P as the sum

$$2P = A - B \tag{22a}$$

where

$$A = \sum_{m=1}^{\infty} m^3 W_m C_{m-r} \tag{22b}$$

$$B = \sum_{m=1}^{\infty} m^3 W_m C_{m+r} \quad . \tag{22c}$$

In Eq. (22b), we let $q = m-r$. Then

$$A = \sum_{q=-r+1}^{\infty} (q + r)^3 W_{q+r} C_q ,$$

or,

$$A = \sum_{q=0}^{\infty} (q + r)^3 W_{q+r} C_q + \sum_{q=-r+1}^{-1} (q + r)^3 W_{q+r} C_q \quad . \tag{22d}$$

We let $q = -p$ in the second term on the right-hand side of Eq. (22d). Thus

$$\sum_{q=-r+1}^{-1} (q + r)^3 W_{q+r} C_q = \sum_{p=r-1}^{1} (r - p)^3 W_{r-p} C_p ,$$

or,

$$\sum_{q=-r+1}^{-1} (q + r)^3 W_{q+r} C_q = \sum_{q=1}^{r-1} (r - q)^3 W_{r-q} C_q \quad . \tag{22e}$$

The upper limit for q on the right-hand side of Eq. (22e) can be changed to ∞ if we make the agreement that Fourier coefficients W_{r-q} with negative subscripts vanish. Hence

$$\sum_{q=-r+1}^{-1} (q + r)^3 W_{q+r} C_q = \sum_{q=1}^{\infty} (r - q)^3 W_{r-q} C_q \tag{22f}$$

Insertion of Eq. (22f) into Eq. (22d) results in the following expression for A

$$A = \sum_{m=0}^{\infty} (m + r)^3 W_{m+r} C_m + \sum_{m=1}^{\infty} (r - m)^3 W_{r-m} C_m$$

$$+ r^3 W_r C_o - r^3 W_r C_o \tag{22g}$$

where the terms $+r^3 W_r C_o$ and $-r^3 W_r C_o$ were added to the right-hand side of Eq. (22d).

But

$$\sum_{m=1}^{\infty} (r - m)^3 W_{r-m} C_m + r^3 W_r C_o = \sum_{m=0}^{\infty} (r - m)^3 W_{r-m} C_m$$

and

$$-r^3 W_r C_o = - \sum_{m=0}^{\infty} \delta_{mo} (m + r)^3 W_{m+r} C_m$$

where δ_{mo} is the Kronecker delta:

$$\delta_{mo} = \begin{cases} 1, m = 0 \\ 0, m \neq 0 \end{cases} .$$

Hence, Eq. (22g) may be written as

$$A = \sum_{m=0}^{\infty} \left[(1 - \delta_{mo})(m + r)^3 W_{m+r} + (r - m)^3 W_{r-m} \right] C_m . \tag{22h}$$

By proceeding in the same way, we can write B in Eq. (22c) as:

$$B = \sum_{q=0}^{\infty} (q - r)^3 W_{q-r} C_q \quad . \tag{22i}$$

We now insert the expressions for A and B given by Eqs. (22h and i) into Eq. (22a) to get

$$2P = \sum_{m=0}^{\infty} \Big[-(m - r)^3 W_{m-r} + (1 - \delta_{mo}) (m + r)^3 W_{m+r}$$

$$+ (r - m)^3 W_{r-m} \Big] C_m \quad . \tag{22j}$$

Then, with the notation

$$\epsilon_{mr} = \begin{cases} +1, & m<r \\ 0, & m=r \\ -1, & m>r \end{cases} ,$$

we can write

$$-(m - r)^3 W_{m-r} + (r - m)^3 W_{r-m} = \epsilon_{mr} |m - r|^3 W_{|m - r|} \quad . \tag{22k}$$

Then from Eqs. (22j and 22k) we obtain, finally,

$$S_r \sum_{m=1}^{\infty} m^3 W_m S_m = \frac{1}{2} \sum_{m=0}^{\infty} \Big[\epsilon_{mr} |m - r|^3 W_{|m-r|}$$

$$+ (1 - \delta_{mo}) (m + r)^3 W_{m+r} \Big] C_m \tag{23}$$

Thus, the term $S_r \sum_{m=1}^{\infty} m^3 W_m S_m$ has been expressed in the form $\sum_{m=0}^{\infty} \left| \quad \right|_m C_m$. By proceeding in the same way, identities may be obtained which enable us to write each term in Eqs. (21) in the form

$$\sum_{m=0,1}^{\infty} \left| \quad \right|_m C_m \text{ , or, } \sum_{m=1,2}^{\infty} \left| \quad \right|_m S_m$$

These identities are:

$$2 C_r \sum_{m=0}^{\infty} W_m C_m = \sum_{m=0}^{\infty} \Big[(1 + \delta_{mr}) W_{|m-r|} + (1 - \delta_{mo}) W_{m+r} \Big] C_m \tag{24a}$$

$$2 S_r \sum_{m=1}^{\infty} V_m S_m = \sum_{m=0}^{\infty} \Big[+ \epsilon_{mr} V_{|m-r|} + (1 - \delta_{mo}) V_{m+r} \Big] C_m \tag{24b}$$

$$2 S_r \sum_{m=0}^{\infty} W_m C_m = \sum_{m=1}^{\infty} \Big[(1 + \delta_{mr}) W_{|m-r|} - W_{m+r} \Big] S_m \tag{24c}$$

$$2 C_r \sum_{m=1}^{\infty} V_m S_m = \sum_{m=1}^{\infty} \Big[- \epsilon_{mr} V_{|m-r|} + V_{m+r} \Big] S_m \tag{24d}$$

$$2 C_r \sum_{m=0}^{\infty} m V_m C_m = \sum_{m=0}^{\infty} \Big[|m-r| V_{|m-r|} + (1 - \delta_{mo}) (m + r) V_{m+r} \Big] C_m \tag{24e}$$

$$\begin{aligned} 2 S_r \sum_{m=1}^{\infty} m W_m S_m = \sum_{m=0}^{\infty} \Big[& + \epsilon_{mr} |m - r| W_{|m-r|} \\ & + (1 - \delta_{mo}) (m + r) W_{m+r} \Big] C_m \end{aligned} \tag{24f}$$

$$2\,S_r \sum_{m=0}^{\infty} m\, V_m\, C_m = \sum_{m=1}^{\infty} \Big[|m-r|\, V_{|m-r|} - (m+r)\, V_{m+r} \Big] S_m \tag{24g}$$

$$2\,C_r \sum_{m=1}^{\infty} m\, W_m\, S_m = \sum_{m=1}^{\infty} \Big[-\epsilon_{mr}\, |m-r|\, W_{|m-r|} + (m+r)\, W_{m+r} \Big] S_m \tag{24h}$$

$$2\,C_r \sum_{m=0}^{\infty} m^2\, W_m\, C_m = \sum_{m=0}^{\infty} \Big[|m-r|^2\, W_{|m-r|} + (1-\delta_{mo})\,(m+r)^2\, W_{m+r} \Big] C_m \tag{24i}$$

$$2\,S_r \sum_{m=1}^{\infty} m^2\, V_m\, S_m = \sum_{m=0}^{\infty} \Big[+\epsilon_{mr}\, |m-r|^2\, V_{|m-r|} + (1-\delta_{mo})\,(m+r)^2\, V_{m+r} \Big] C_m \tag{24j}$$

$$2\,C_r \sum_{m=1}^{\infty} m^2\, V_m\, S_m = \sum_{m=1}^{\infty} \Big[-\epsilon_{mr}\, |m-r|^2\, V_{|m-r|} + (m+r)^2\, V_{m+r} \Big] S_m \tag{24k}$$

$$2\,S_r \sum_{m=0}^{\infty} m^2\, W_m\, C_m = \sum_{m=1}^{\infty} \Big[|m-r|^2\, W_{|m-r|} - (m+r)^2\, W_{m+r} \Big] S_m \tag{24l}$$

$$2\,C_r \sum_{m=0}^{\infty} m^3\, V_m\, C_m = \sum_{m=0}^{\infty} \Big[|m-r|^3\, V_{|m-r|} + (1-\delta_{mo})\,(m+r)^3\, V_{m+r} \Big] C_m \tag{24m}$$

$$2\,S_r \sum_{m=1}^{\infty} m^3\, W_m\, S_m = \sum_{m=0}^{\infty} \Big[+\epsilon_{mr}\, |m-r|^3\, W_{|m-r|} + (1-\delta_{mo})\,(m+r)^3\, W_{m+r} \Big] C_m \tag{24n}$$

$$2\, C_r \sum_{m=1}^{\infty} m^3\, W_m\, S_m \;=\; \sum_{m=1}^{\infty} \Big[-\epsilon_{mr}\, |m - r|^3\, W_{|m-r|} + (m + r)^3\, W_{m+r} \Big] S_m \tag{24o}$$

$$2\, S_r \sum_{m=0}^{\infty} m^3\, V_m\, C_m \;=\; \sum_{m=1}^{\infty} \Big[|m - r|^3\, V_{|m-r|} - (m + r)^3\, V_{m+r} \Big] S_m \tag{24p}$$

$$2\, C_r \sum_{m=1}^{\infty} m^4\, V_m\, S_m \;=\; \sum_{m=1}^{\infty} \Big[-\epsilon_{mr}\, |m - r|^4\, V_{|m-r|} + (m + r)^4\, V_{m+r} \Big] S_m \tag{24q}$$

$$2\, C_r \sum_{m=0}^{\infty} m^4\, W_m\, C_m \;=\; \sum_{m=0}^{\infty} \Big[|m - r|^4\, W_{|m-r|} + (1 - \delta_{mo})\, (m + r)^4\, W_{m+r} \Big] C_m \tag{24r}$$

where

$$\delta_{mr} = \begin{cases} 1, & m=r \\ 0, & m\neq r \end{cases} \quad , \quad r = 1, \ldots, 4 \quad , \tag{25}$$

$$\delta_{mo} = \begin{cases} 1, & m=0 \\ 0, & m\neq 0 \end{cases} \quad , \tag{26}$$

$$\epsilon_{mr} = \begin{cases} 1, & m<r \\ 0, & m=r \\ -1, & m>r \end{cases} \quad , \quad r = 1, \ldots, 4 \quad . \tag{27}$$

We note that identities involving U_m may be obtained by replacing W_m in Eqs. (24) by U_m .

We now apply the transformations given by Eqs. (24) to the stability equations [Eqs. (21)]. As a result, we find that all terms in each of Eqs. (21) may be written as an infinite summation over the same trigonometric function and over the same range for m . Hence, in each of Eqs. (21), we may collect all

terms under the same summation sign and arrive at the following form of the stability equations for buckling Mode A:

$$\sum_{m=0}^{\infty} \Bigg\{ 2\,(h_{10} + c_{10}\lambda)\, U_m + \sum_{r=1}^{3} (h_{1r} + c_{1r}\lambda) \Big[(1 + \delta_{mr})\, U_{|m-r|}$$

$$+ (1 - \delta_{mo})\, U_{m+r} \Big] - \sum_{r=1}^{3} h_{2r} \Big[\epsilon_{mr}\, |m - r|\, U_{|m-r|}$$

$$+ (1 - \delta_{mo})\, (m + r)\, U_{m+r} \Big] - 2m^2\, h_{30}\, U_m$$

$$- \sum_{r=1}^{3} h_{3r} \Big[|m - r|^2\, U_{|m-r|} + (1 - \delta_{mo})\, (m + r)^2\, U_{m+r} \Big]$$

$$+ \sum_{r=1}^{4} (h_{4r} + c_{4r}\lambda) \Big[\epsilon_{mr}\, V_{|m-r|} + (1 - \delta_{mo})\, V_{m+r} \Big]$$

$$+ 2m\,(h_{50} + c_{50}\lambda)\, V_m + \sum_{r=1}^{4} (h_{5r} + c_{5r}\lambda) \Big[|m - r|\, V_{|m-r|}$$

$$+ (1 - \delta_{mo})\, (m + r)\, V_{m+r} \Big] - \sum_{r=1}^{4} h_{6r} \Big[\epsilon_{mr}\, |m - r|^2\, V_{|m-r|}$$

$$+ (1 - \delta_{mo})\, (m + r)^2\, V_{m+r} \Big] - 2m^3\, h_{70}\, V_m$$

$$- \sum_{r=1}^{4} h_{7r} \Big[|m - r|^3\, V_{|m-r|} + (1 - \delta_{mo})\, (m + r)^3\, V_{m+r} \Big]$$

$$+ 2\,(h_{80} + c_{80}\lambda)\, W_m + \sum_{r=1}^{4} (h_{8r} + c_{8r}\lambda) \Big\{ (1 + \delta_{mr})\, W_{|m-r|}$$

$$+ (1 - \delta_{mo})\, W_{m+r} \Big]$$

$$- \sum_{r=1}^{4} (h_{9r} + c_{9r}\lambda) \left[\epsilon_{mr} |m - r| W_{|m-r|} + (1 - \delta_{mo}) (m + r) W_{m+r} \right]$$

$$- 2m^2 (h_{10,0} + c_{10,0}\lambda) W_m - \sum_{r=1}^{4} (h_{10,r} + c_{10,r}\lambda) \left[|m - r|^2 W_{|m-r|} \right.$$

$$\left. + (1 - \delta_{mo}) (m + r)^2 W_{m+r} \right] + \sum_{r=1}^{4} h_{11,r} \left[\epsilon_{mr} |m - r|^3 W_{|m-r|} \right.$$

$$\left. + (1 - \delta_{mo}) (m + r)^3 W_{m+r} \right] + 2m^4 h_{12,0} W_m + \sum_{r=1}^{4} h_{12,r} \left[|m - r|^4 W_{|m-r|} \right.$$

$$\left. \left. + (1 - \delta_{mo}) (m + r)^4 W_{m+r} \right] \right\} \cos (m\psi) = 0 \qquad (28a)$$

$$\sum_{m=1}^{\infty} \left\{ f_{11} \left[(1 + \delta_{m1}) U_{|m-1|} - U_{m+1} \right] - 2m f_{20} U_m - f_{21} \left[-\epsilon_{m1} |m - 1| U_{|m-1|} \right. \right.$$

$$\left. + (m + 1) U_{m+1} \right] + 2 (f_{40} + a_{40}\lambda) V_m$$

$$+ \sum_{r-1}^{2} (f_{4r} + a_{4r}\lambda) \left[-\epsilon_{mr} V_{|m-r|} + V_{m+r} \right]$$

$$+ \sum_{r-1}^{2} f_{5r} \left[|m - r| V_{|m-r|} - (m + r) V_{m+r} \right] - 2m^2 f_{60} V_m$$

$$- \sum_{r=1}^{2} f_{6r} \left[-\epsilon_{mr} |m - r|^2 V_{|m-r|} + (m + r)^2 V_{m+r} \right]$$

$$+ f_{81} \left[(1 + \delta_{m1}) W_{|m-1|} - W_{m+1} \right] - 2m (f_{90} + a_{90}\lambda) W_m$$

$$- \sum_{r=1}^{2} (f_{9r} + a_{9r}\lambda) \left[-\epsilon_{mr} |m - r| W_{|m-r|} \right.$$

$$\left. \left. + (m + r) W_{m+r} \right] \right\} \sin (m\psi) = 0 \qquad (28b)$$

$$\sum_{m=0}^{\infty} \left\{ 2(g_{10} + b_{10}\lambda)\, U_m + \sum_{r=1}^{3} (g_{1r} + b_{1r}\lambda) \left[(1 + \delta_{mr}) U_{|m-r|} + (1 - \delta_{mo})\, U_{m+r} \right] \right.$$

$$- \sum_{r=1}^{3} (g_{2r} + b_{2r}\lambda) \left[\epsilon_{mr}\, |m - r|\, U_{|m-r|} \right.$$

$$\left. + (1 - \delta_{mo})\,(m + r)\, U_{m+r} \right] - 2m^2 (g_{30} + b_{30}\lambda)\, U_m$$

$$- \sum_{r=1}^{3} (g_{3r} + b_{3r}\lambda) \left[|m - r|^2 U_{|m-r|} + (1 - \delta_{mo})\,(m + r)^2 U_{m+r} \right]$$

$$+ \sum_{r=1}^{2} (g_{4r} + b_{4r}\lambda) \left[\epsilon_{mr} V_{|m-r|} + (1 - \delta_{mo})\, V_{m+r} \right]$$

$$+ 2m\, g_{50} V_m + \sum_{r=1}^{2} g_{5r} \left[|m - r|\, V_{|m-r|} + (1 - \delta_{mo})\,(m + r)\, V_{m+r} \right]$$

$$+ 2\,(g_{80} + b_{80}\lambda)\, W_m + \sum_{r=1}^{2} (g_{8r} + b_{8r}\lambda) \left[(1 + \delta_{mr})\, W_{|m-r|} \right.$$

$$\left. \left. + (1 - \delta_{mo})\, W_{m+r} \right] \right\} \cos(m\psi) = 0 \qquad (28c)$$

In order for Eqs. (28) to be identically satisfied for all values of ψ, each of the braced expressions in Eqs. (28) must vanish. This yields

$$z_{10}^{(m)} U_m + \sum_{r=1}^{3} z_{2r}^{(m)} U_{|m-r|} + \sum_{r=1}^{3} z_{3r}^{(m)} U_{m+r} + z_{40}^{(m)} V_m + \sum_{r=1}^{4} z_{5r}^{(m)} V_{|m-r|}$$

$$+ \sum_{r=1}^{4} z_{6r}^{(m)} V_{m+r} + z_{70}^{(m)} W_m + \sum_{r=1}^{4} z_{8r}^{(m)} W_{|m-r|}$$

$$+ \sum_{r=1}^{4} z_{9r}^{(m)} W_{m+r} = 0 \,, \quad m = (0, 1, 2, \ldots.) \qquad (29a)$$

$$
x_{10}^{(m)} U_m + x_{21}^{(m)} U_{|m-1|} + x_{31}^{(m)} U_{m+1} + x_{40}^{(m)} V_m
$$

$$
+ \sum_{r=1}^{2} x_{5r}^{(m)} V_{|m-r|} + \sum_{r=1}^{2} x_{6r}^{(m)} V_{m+r} + x_{70}^{(m)} W_m
$$

$$
+ \sum_{r=1}^{2} x_{8r}^{(m)} W_{|m-r|} + \sum_{r=1}^{2} x_{9r}^{(m)} W_{m+r} = 0 \, , \quad m = (1,2,\ldots) \tag{29b}
$$

$$
y_{10}^{(m)} U_m + \sum_{r=1}^{3} y_{2r}^{(m)} U_{|m-r|} + \sum_{r=1}^{3} y_{3r}^{(m)} U_{m+r} + y_{40}^{(m)} V_m + \sum_{r=1}^{2} y_{5r}^{(m)} V_{|m-r|}
$$

$$
+ \sum_{r=1}^{2} y_{6r}^{(m)} V_{m+r} + y_{70}^{(m)} W_m + \sum_{r=1}^{2} y_{8r}^{(m)} W_{|m-r|}
$$

$$
+ \sum_{r=1}^{2} y_{9r}^{(m)} W_{m+r} = 0 \, , \quad m = (0,1,2,\ldots) \tag{29c}
$$

where

$$
\begin{aligned}
z_{ij}^{(m)} &= \hat{z}_{ij}^{(m)} - \omega \bar{z}_{ij}^{(m)} \\
x_{ij}^{(m)} &= \hat{x}_{ij}^{(m)} - \omega \bar{x}_{ij}^{(m)} \\
y_{ij}^{(m)} &= \hat{y}_{ij}^{(m)} - \omega \bar{y}_{ij}^{(m)} \, ,
\end{aligned} \tag{30}
$$

and

$$
\omega = \frac{1}{\lambda} = \frac{1}{pa/Eh} \, . \tag{31}
$$

The coefficients on the right-hand sides of Eqs. (30) are given by

$$\bar{z}_{10}^{(m)} = -\left[2 h_{10} - 2m^2 h_{30}\right] \tag{32}$$

$$\bar{z}_{2r}^{(m)} = -\left[(1 + \delta_{mr}) h_{1r} - |m - r| \epsilon_{mr} h_{2r} - |m - r|^2 h_{3r}\right], \quad (r = 1,2,3)$$

$$\bar{z}_{3r}^{(m)} = -(1 - \delta_{mo}) \left[h_{1r} - (m + r) h_{2r} - (m + r)^2 h_{3r}\right], \quad (r = 1,2,3)$$

$$\bar{z}_{40}^{(m)} = -\left[2m\, h_{50} - 2m^3 h_{70}\right]$$

$$\bar{z}_{5r}^{(m)} = -\left[\epsilon_{mr} h_{4r} + |m - r| h_{5r} - |m - r|^2 \epsilon_{mr} h_{6r} - |m - r|^3 h_{7r}\right], \quad (r = 1,2,3,4)$$

$$\bar{z}_{6r}^{(m)} = -(1 - \delta_{mo}) \left[h_{4r} + (m + r) h_{5r} - (m + r)^2 h_{6r} - (m + r)^3 h_{7r}\right], \quad (r = 1,2,3,4)$$

$$\bar{z}_{70}^{(m)} = -\left[2 h_{80} - 2m^2 h_{10,0} + 2m^4 h_{12,0}\right]$$

$$\bar{z}_{8r}^{(m)} = -\left[(1 + \delta_{mr}) h_{8r} - |m-r| \epsilon_{mr} h_{9r} - |m - r|^2 h_{10,r} + |m - r|^3 \epsilon_{mr} h_{11,r} + |m - r|^4 h_{12,r}\right], \quad (r = 1,2,3,4)$$

$$\bar{z}_{9r}^{(m)} = -(1 - \delta_{mo}) \left[h_{8r} - (m + r) h_{9r} - (m + r)^2 h_{10,r} + (m + r)^3 h_{11,r} + (m + r)^4 h_{12,r}\right], \quad (r = 1,2,3,4)$$

$$\hat{z}_{10}^{(m)} = 2\, c_{10}$$

$$\hat{z}_{2r}^{(m)} = (1 + \delta_{mr})\, c_{1r}, \qquad (r = 1, 2, 3)$$

$$\hat{z}_{3r}^{(m)} = (1 - \delta_{mo})\, c_{1r}, \qquad (r = 1, 2, 3)$$

$$\hat{z}_{40}^{(m)} = 2m\, c_{50} \tag{32 cont'd}$$

$$\hat{z}_{5r}^{(m)} = \epsilon_{mr}\, c_{4r} + |m - r|\, c_{5r} \quad (r = 1,2,3,4)$$

$$\hat{z}_{6r}^{(m)} = (1 - \delta_{mo}) \left[c_{4r} + (m + r)\, c_{5r} \right] \quad (r = 1,2,3,4)$$

$$\hat{z}_{70}^{(m)} = 2\, c_{80} - 2m^2\, C_{10,0}$$

$$\hat{z}_{8r}^{(m)} = (1 + \delta_{mr})\, c_{8r} - |m - r|\, \epsilon_{mr}\, c_{9r} - |m - r|^2\, c_{10,r} \quad (r = 1,2,3,4)$$

$$\hat{z}_{9r}^{(m)} = (1 - \delta_{mo}) \left[c_{8r} - (m + r)\, c_{9r} - (m + r)^2\, c_{10,r} \right], (r = 1,2,3,4)$$

$$\bar{x}_{10}^{(m)} = -\left[-2m\, f_{20} \right]$$

$$\bar{x}_{21}^{(m)} = -\left[(1 + \delta_{m1})\, f_{11} + |m - 1|\, \epsilon_{m1}\, f_{21} \right]$$

$$\bar{x}_{31}^{(m)} = -\left[-f_{11} - (m + 1)\, f_{21} \right]$$

$$\bar{x}_{40}^{(m)} = -\left[2\, f_{40} - 2m^2\, f_{60} \right]$$

$$\bar{x}_{5r}^{(m)} = -\left[-\epsilon_{mr}\, f_{4r} + |m - r|\, f_{5r} + |m - r|^2\, \epsilon_{mr}\, f_{6r} \right], \quad (r = 1,2)$$

$$\bar{x}_{6r}^{(m)} = -\left[f_{4r} - (m + r)\, f_{5r} - (m + r)^2\, f_{6r} \right], \quad (r = 1,2)$$

$$\bar{x}_{70}^{(m)} = -\left[-2m\, f_{90} \right]$$

$$\bar{x}_{8r}^{(m)} = -\left[(1 + \delta_{mr})\, \delta_{r1}\, f_{8r} + |m - r|\, \epsilon_{mr}\, f_{9r} \right], \quad (r = 1, 2)$$

$$\bar{x}_{9r}^{(m)} = -\left[-\delta_{r1}\, f_{8r} - (m + r)\, f_{9r} \right], \quad (r = 1, 2)$$

$$\hat{x}_{10}^{(m)} = 0$$

$$\hat{x}_{21}^{(m)} = 0$$

$$\hat{x}_{31}^{(m)} = 0 \qquad \text{(32 cont'd)}$$

$$\hat{x}_{40}^{(m)} = 2\, a_{40}$$

$$\hat{x}_{5r}^{(m)} = -\epsilon_{mr}\, a_{4r} \quad (r = 1, 2)$$

$$\hat{x}_{6r}^{(m)} = a_{4r} \quad (r = 1, 2)$$

$$\hat{x}_{70}^{(m)} = -2m\, a_{90}$$

$$\hat{x}_{8r}^{(m)} = |m - r|\, \epsilon_{mr}\, a_{9r} \quad (r = 1, 2)$$

$$\hat{x}_{9r}^{(m)} = -(m + r)\, a_{9r} \quad (r = 1, 2)$$

$$\bar{y}_{10}^{(m)} = -\left[2\, g_{10} - 2m^2\, g_{30}\right]$$

$$\bar{y}_{2r}^{(m)} = -\left[(1 + \delta_{mr})\, g_{1r} - |m - r|\, \epsilon_{mr}\, g_{2r} - |m - r|^2\, g_{3r}\right], \quad (r = 1, 2, 3)$$

$$\bar{y}_{3r}^{(m)} = -(1 - \delta_{mo})\left[g_{1r} - (m + r)\, g_{2r} - (m + r)^2\, g_{3r}\right], \quad (r = 1, 2, 3)$$

$$\bar{y}_{40}^{(m)} = -\left[2m\, g_{50}\right]$$

$$\bar{y}_{5r}^{(m)} = -\left[\epsilon_{mr}\, g_{4r} + |m - r|\, g_{5r}\right], \quad (r = 1, 2)$$

$$\bar{y}_{6r}^{(m)} = -(1 - \delta_{mo})\left[g_{4r} + (m + r)\, g_{5r}\right], \quad (r = 1, 2)$$

$$\bar{y}_{70}^{(m)} = -2\left[g_{80}\right]$$

$$\bar{y}_{8r}^{(m)} = -(1 + \delta_{mr})\, g_{8r} \quad (r = 1, 2)$$

$$\bar{y}_{9r}^{(m)} = -(1 - \delta_{mo})\, g_{8r} \quad (r = 1, 2)$$

$$\hat{y}_{10}^{(m)} = 2\, b_{10} - 2m^2\, b_{30}$$

$$\hat{y}_{2r}^{(m)} = (1 + \delta_{mr})\, b_{1r} - |m - r|\, \epsilon_{mr}\, b_{2r} - |m - r|^2\, b_{3r} \; , \quad (r = 1,2,3)$$

(32 concl'd)

$$\hat{y}_{3r}^{(m)} = (1 - \delta_{mo})\left[b_{1r} - (m + r)\, b_{2r} - (m + r)^2\, b_{3r}\right] \quad (r = 1,2,3)$$

$$\hat{y}_{40}^{(m)} = 0$$

$$\hat{y}_{5r}^{(m)} = \epsilon_{mr}\, b_{4r} \quad (r = 1,2)$$

$$\hat{y}_{6r}^{(m)} = (1 - \delta_{mo})\, b_{4r} \quad (r = 1,2)$$

$$\hat{y}_{70}^{(m)} = 2\, b_{80}$$

$$\hat{y}_{8r}^{(m)} = (1 + \delta_{mr})\, b_{8r} \quad (r = 1,2)$$

$$\hat{y}_{9r}^{(m)} = (1 - \delta_{mo})\, b_{8r} \quad (r = 1,2)$$

The coefficients a , b , c , f , g , and h in Eqs. (32) are given by Eqs. (15).

By letting m take on the values $m = 0, 1, 2 \ldots.$ in Eqs. (29), we obtain an infinite system of algebraic equations in which the unknowns are the Fourier coefficients U_m , V_m , and W_m . The coefficients in this system of equations are shown in Table 1. Hence, the system of ordinary differential equations with variable coefficients [Eqs. (10 and 11)] has been replaced by an infinite system of linear homogeneous algebraic equations [Eqs. (29), or, see Table 1] for U_m , V_m , and W_m .

Using matrix notation, we rewrite Eqs. (29) as:

$$\left[R\right]\left\{V\right\} - \omega\left[S\right]\left\{V\right\} = \left\{0\right\} \tag{33}$$

where $\left[R\right]$ and $\left[S\right]$ are square matrices formed by the coefficients $\left(\hat{x}_{ij}^{(m)}, \hat{y}_{ij}^{(m)}, \hat{z}_{ij}^{(m)}\right)$ and $\left(\bar{x}_{ij}^{(m)}, \bar{y}_{ij}^{(m)}, \bar{z}_{ij}^{(m)}\right)$, respectively; $\left\{V\right\}$ is a

column vector formed by the unknown Fourier coefficients U_m, V_m, W_m. The elements $r_{k,\ell}$ and $s_{k,\ell}$ of [R] and [S] can be obtained from Table 1 and Eqs. (30, 32, and 15). For example, the diagonal element $s_{5,5}$ in the 5^{th} row of [S] may be expressed in terms of the geometric parameters $\frac{b}{a}$ and $\frac{h}{a}$, and the number of circumferential waves n, as follows:

$$s_{5,5} \overset{\text{Table 1 \& Eqs. (30)}}{=\!=\!=\!=\!=\!=} \bar{z}_{70}^{(1)} + \bar{z}_{82}^{(1)}$$

$$\overset{\text{Eqs. (32)}}{=\!=\!=} - 2\, h_{80} - h_{82} + h_{92} + 2h_{10,0} + h_{10,2} - h_{11,2} - 2h_{12,0} - h_{12,2}$$

$$\overset{\text{Eqs. (15)}}{=\!=\!=} \frac{1}{12}\left[-3 - \nu - (12 + 3\nu)\left(\frac{b}{a}\right)^2 - 2\left(\frac{b}{a}\right)^4 + \left(\frac{5}{2} - \frac{3}{2}\nu - 2n^2\right)n^2 - 4n^2\left(\frac{b}{a}\right)^2\right]\left(\frac{h}{a}\right)^2 - \frac{5}{2}(1 + \nu) - \left(\frac{21}{2} + 9\nu\right)\left(\frac{b}{a}\right)^2 - 2\left(\frac{b}{a}\right)^4$$

where the reference above the long equal sign indicates the means by which the right-hand side was obtained. Similarly the element $r_{5,5} = \hat{z}_{70}^{(1)} + \hat{z}_{82}^{(1)}$.

For fixed values of the geometric parameters $\alpha = \frac{b}{a}$ and $k = \frac{1}{12}\left(\frac{h}{a}\right)^2$, and for a fixed value of the number of circumferential waves n, the coefficients $z_{ij}^{(m)}$, $x_{ij}^{(m)}$, and $y_{ij}^{(m)}$ in Eqs. (33) depend only on the load parameter λ. The objective of the stability analysis is to determine the lowest value of λ (or equivalently, the highest value of $\omega = 1/\lambda$) for which Eqs. (33) admit a non-trivial solution. This value of λ is called an eigenvalue of the system of equations and the associated nontrivial solution is called an eigenvector. The components of the eigenvector are the Fourier coefficients U_m, V_m, and W_m. These components are only determined up to a scalar factor. The displacement functions

$$u_n(\psi) = \sum_{m=0}^{\infty} U_m \cos m\psi \tag{34a}$$

	Coefficients of:	U_o	W_o	U_1	V_1	W_1	U_2	V_2	W_2	U_3	V_3	W_3	U_4	V_4	W_4
		1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	m = 0	$y_{10}^{(0)}$	$y_{70}^{(0)}$	$y_{21}^{(0)}$	$y_{51}^{(0)}$	$y_{81}^{(0)}$	$y_{22}^{(0)}$	$y_{52}^{(0)}$	$y_{82}^{(0)}$	$y_{23}^{(0)}$					
2		$z_{10}^{(0)}$	$z_{70}^{(0)}$	$z_{21}^{(0)}$	$z_{51}^{(0)}$	$z_{81}^{(0)}$	$z_{22}^{(0)}$	$z_{52}^{(0)}$	$z_{82}^{(0)}$	$z_{23}^{(0)}$	$z_{53}^{(0)}$	$z_{83}^{(0)}$		$z_{54}^{(0)}$	$z_{84}^{(0)}$
3		$x_{21}^{(1)}$	$x_{81}^{(1)}$	$x_{10}^{(1)}$	$x_{40}^{(1)}+x_{52}^{(1)}$	$x_{70}^{(1)}+x_{82}^{(1)}$	$x_{31}^{(1)}$	$x_{61}^{(1)}$	$x_{91}^{(1)}$		$x_{62}^{(1)}$	$x_{92}^{(1)}$			
4	m = 1	$y_{21}^{(1)}$	$y_{81}^{(1)}$	$y_{10}^{(1)}+y_{22}^{(1)}$	$y_{40}^{(1)}+y_{52}^{(1)}$	$y_{70}^{(1)}+y_{82}^{(1)}$	$y_{23}^{(1)}+y_{31}^{(1)}$	$y_{61}^{(1)}$	$y_{91}^{(1)}$	$y_{32}^{(1)}$	$y_{62}^{(1)}$	$y_{92}^{(1)}$	$y_{33}^{(1)}$		
5		$z_{21}^{(1)}$	$z_{81}^{(1)}$	$z_{10}^{(1)}+z_{22}^{(1)}$	$z_{40}^{(1)}+z_{52}^{(1)}$	$z_{70}^{(1)}+z_{82}^{(1)}$	$z_{23}^{(1)}+z_{31}^{(1)}$	$z_{53}^{(1)}+z_{61}^{(1)}$	$z_{83}^{(1)}+z_{91}^{(1)}$	$z_{32}^{(1)}$	$z_{54}^{(1)}+z_{62}^{(1)}$	$z_{84}^{(1)}+z_{92}^{(1)}$	$z_{33}^{(1)}$	$z_{63}^{(1)}$	$z_{93}^{(1)}$
6			$x_{82}^{(2)}$	$x_{21}^{(2)}$	$x_{51}^{(2)}$	$x_{81}^{(2)}$	$x_{10}^{(2)}$	$x_{40}^{(2)}$	$x_{70}^{(2)}$	$x_{31}^{(2)}$	$x_{61}^{(2)}$	$x_{91}^{(2)}$		$x_{62}^{(2)}$	$x_{92}^{(2)}$
7	m = 2	$y_{22}^{(2)}$	$y_{82}^{(2)}$	$y_{21}^{(2)}+y_{23}^{(2)}$	$y_{51}^{(2)}$	$y_{81}^{(2)}$	$y_{10}^{(2)}$	$y_{40}^{(2)}$	$y_{70}^{(2)}$	$y_{31}^{(2)}$	$y_{61}^{(2)}$	$y_{91}^{(2)}$	$y_{32}^{(2)}$	$y_{62}^{(2)}$	$y_{92}^{(2)}$
8		$z_{22}^{(2)}$	$z_{82}^{(2)}$	$z_{21}^{(2)}+z_{23}^{(2)}$	$z_{51}^{(2)}+z_{53}^{(2)}$	$z_{81}^{(2)}+z_{83}^{(2)}$	$z_{10}^{(2)}$	$z_{40}^{(2)}+z_{54}^{(2)}$	$z_{70}^{(2)}+z_{84}^{(2)}$	$z_{31}^{(2)}$	$z_{61}^{(2)}$	$z_{91}^{(2)}$	$z_{32}^{(2)}$	$z_{62}^{(2)}$	$z_{92}^{(2)}$
9					$x_{52}^{(3)}$	$x_{82}^{(3)}$	$x_{21}^{(3)}$	$x_{51}^{(3)}$	$x_{81}^{(3)}$	$x_{10}^{(3)}$	$x_{40}^{(3)}$	$x_{70}^{(3)}$	$x_{31}^{(3)}$	$x_{61}^{(3)}$	$x_{91}^{(3)}$
10	m = 3	$y_{23}^{(3)}$		$y_{22}^{(3)}$	$y_{52}^{(3)}$	$y_{82}^{(3)}$	$y_{21}^{(3)}$	$y_{51}^{(3)}$	$y_{81}^{(3)}$	$y_{10}^{(3)}$	$y_{40}^{(3)}$	$y_{70}^{(3)}$	$y_{31}^{(3)}$	$y_{61}^{(3)}$	$y_{91}^{(3)}$
11		$z_{23}^{(3)}$	$z_{83}^{(3)}$	$z_{22}^{(3)}$	$z_{52}^{(3)}+z_{54}^{(3)}$	$z_{82}^{(3)}+z_{84}^{(3)}$	$z_{21}^{(3)}$	$z_{51}^{(3)}$	$z_{81}^{(3)}$	$z_{10}^{(3)}$	$z_{40}^{(3)}$	$z_{70}^{(3)}$	$z_{31}^{(3)}$	$z_{61}^{(3)}$	$z_{91}^{(3)}$
12								$x_{52}^{(4)}$	$x_{82}^{(4)}$	$x_{21}^{(4)}$	$x_{51}^{(4)}$	$x_{81}^{(4)}$	$x_{10}^{(4)}$	$x_{40}^{(4)}$	$x_{70}^{(4)}$
13	m = 4			$y_{23}^{(4)}$			$y_{22}^{(4)}$	$y_{52}^{(4)}$	$y_{82}^{(4)}$	$y_{21}^{(4)}$	$y_{51}^{(4)}$	$y_{81}^{(4)}$	$y_{10}^{(4)}$	$y_{40}^{(4)}$	$y_{70}^{(4)}$
14			$z_{84}^{(4)}$	$z_{23}^{(4)}$	$z_{53}^{(4)}$	$z_{83}^{(4)}$	$z_{22}^{(4)}$	$z_{52}^{(4)}$	$z_{82}^{(4)}$	$z_{21}^{(4)}$	$z_{51}^{(4)}$	$z_{81}^{(4)}$	$z_{10}^{(4)}$	$z_{40}^{(4)}$	$z_{70}^{(4)}$
15											$x_{52}^{(m)}$	$x_{82}^{(m)}$	$x_{21}^{(m)}$	$x_{51}^{(m)}$	$x_{81}^{(m)}$
16	m = 5						$y_{23}^{(m)}$			$y_{22}^{(m)}$	$y_{52}^{(m)}$	$y_{82}^{(m)}$	$y_{21}^{(m)}$	$y_{51}^{(m)}$	$y_{81}^{(m)}$
17					$z_{54}^{(m)}$	$z_{84}^{(m)}$	$z_{23}^{(m)}$	$z_{53}^{(m)}$	$z_{83}^{(m)}$	$z_{22}^{(m)}$	$z_{52}^{(m)}$	$z_{82}^{(m)}$	$z_{21}^{(m)}$	$z_{51}^{(m)}$	$z_{81}^{(m)}$
18														$x_{52}^{(m)}$	$x_{82}^{(m)}$
19	m = 6									$y_{23}^{(m)}$			$y_{22}^{(m)}$	$y_{52}^{(m)}$	$y_{82}^{(m)}$
20								$z_{54}^{(m)}$	$z_{84}^{(m)}$	$z_{23}^{(m)}$	$z_{53}^{(m)}$	$z_{83}^{(m)}$	$z_{22}^{(m)}$	$z_{52}^{(m)}$	$z_{82}^{(m)}$
21															
22	m = 7												$y_{23}^{(m)}$		
23											$z_{54}^{(m)}$	$z_{84}^{(m)}$	$z_{23}^{(m)}$	$z_{53}^{(m)}$	$z_{83}^{(m)}$
.														.	
.														.	
.														.	

THIS PATTERN REPEATS FOR m > 5

Table 1
MODE A STABILITY EQUATIONS

W_4	U_5	V_5	W_5	U_6	V_6	W_6	U_7	V_7	W_7	U_8	V_8	W_8	U_9	V_9	W_9	U_{10}	V_{10}	W_{10}	U_{11}	V_{11}	W_{11}	...	
14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	...	
$z_{84}^{(0)}$																							
$z_{93}^{(1)}$		$z_{64}^{(1)}$	$z_{94}^{(1)}$																				
$x_{92}^{(2)}$																							
$y_{92}^{(2)}$	$y_{32}^{(2)}$																						
$z_{92}^{(2)}$	$z_{33}^{(2)}$	$z_{63}^{(2)}$	$z_{93}^{(2)}$		$z_{64}^{(2)}$	$z_{94}^{(2)}$																	
$x_{91}^{(3)}$		$x_{62}^{(3)}$	$x_{92}^{(3)}$																				
$y_{91}^{(3)}$	$y_{32}^{(3)}$	$y_{62}^{(3)}$	$y_{92}^{(3)}$	$y_{33}^{(3)}$																			
$z_{91}^{(3)}$	$z_{32}^{(3)}$	$z_{62}^{(3)}$	$z_{92}^{(3)}$	$z_{33}^{(3)}$	$z_{63}^{(3)}$	$z_{93}^{(3)}$		$z_{64}^{(3)}$	$z_{94}^{(3)}$														
$x_{70}^{(4)}$	$x_{31}^{(4)}$	$x_{61}^{(4)}$	$x_{91}^{(4)}$		$x_{62}^{(4)}$	$x_{92}^{(4)}$																	
$y_{70}^{(4)}$	$y_{31}^{(4)}$	$y_{61}^{(4)}$	$y_{91}^{(4)}$	$y_{32}^{(4)}$	$y_{62}^{(4)}$	$y_{92}^{(4)}$	$y_{33}^{(4)}$																
$z_{70}^{(4)}$	$z_{31}^{(4)}$	$z_{61}^{(4)}$	$z_{91}^{(4)}$	$z_{32}^{(4)}$	$z_{62}^{(4)}$	$z_{92}^{(4)}$	$z_{33}^{(4)}$	$z_{63}^{(4)}$	$z_{93}^{(4)}$		$z_{64}^{(4)}$	$z_{94}^{(4)}$											
$x_{81}^{(m)}$	$x_{10}^{(m)}$	$x_{40}^{(m)}$	$x_{70}^{(m)}$	$x_{31}^{(m)}$	$x_{61}^{(m)}$	$x_{91}^{(m)}$		$x_{62}^{(m)}$	$x_{92}^{(m)}$														5
$y_{81}^{(m)}$	$y_{10}^{(m)}$	$y_{40}^{(m)}$	$y_{70}^{(m)}$	$y_{31}^{(m)}$	$y_{61}^{(m)}$	$y_{91}^{(m)}$	$y_{32}^{(m)}$	$y_{62}^{(m)}$	$y_{92}^{(m)}$	$y_{33}^{(m)}$													5
$z_{81}^{(m)}$	$z_{10}^{(m)}$	$z_{40}^{(m)}$	$z_{70}^{(m)}$	$z_{31}^{(m)}$	$z_{61}^{(m)}$	$z_{91}^{(m)}$	$z_{32}^{(m)}$	$z_{62}^{(m)}$	$z_{92}^{(m)}$	$z_{33}^{(m)}$	$z_{63}^{(m)}$	$z_{93}^{(m)}$		$z_{64}^{(m)}$	$z_{94}^{(m)}$								5
$x_{82}^{(m)}$	$x_{21}^{(m)}$	$x_{51}^{(m)}$	$x_{81}^{(m)}$	$x_{10}^{(m)}$	$x_{40}^{(m)}$	$x_{70}^{(m)}$	$x_{31}^{(m)}$	$x_{61}^{(m)}$	$x_{91}^{(m)}$		$x_{62}^{(m)}$	$x_{92}^{(m)}$											6
$y_{82}^{(m)}$	$y_{21}^{(m)}$	$y_{51}^{(m)}$	$y_{81}^{(m)}$	$y_{10}^{(m)}$	$y_{40}^{(m)}$	$y_{70}^{(m)}$	$y_{31}^{(m)}$	$y_{61}^{(m)}$	$y_{91}^{(m)}$	$y_{32}^{(m)}$	$y_{62}^{(m)}$	$y_{92}^{(m)}$	$y_{33}^{(m)}$										6
$z_{82}^{(m)}$	$z_{21}^{(m)}$	$z_{51}^{(m)}$	$z_{81}^{(m)}$	$z_{10}^{(m)}$	$z_{40}^{(m)}$	$z_{70}^{(m)}$	$z_{31}^{(m)}$	$z_{61}^{(m)}$	$z_{91}^{(m)}$	$z_{32}^{(m)}$	$z_{62}^{(m)}$	$z_{92}^{(m)}$	$z_{33}^{(m)}$	$z_{63}^{(m)}$	$z_{93}^{(m)}$		$z_{64}^{(m)}$	$z_{94}^{(m)}$					6
		$x_{52}^{(m)}$	$x_{82}^{(m)}$	$x_{21}^{(m)}$	$x_{51}^{(m)}$	$x_{81}^{(m)}$	$x_{10}^{(m)}$	$x_{40}^{(m)}$	$x_{70}^{(m)}$	$x_{31}^{(m)}$	$x_{61}^{(m)}$	$x_{91}^{(m)}$		$x_{62}^{(m)}$	$x_{92}^{(m)}$								7
	$y_{22}^{(m)}$	$y_{52}^{(m)}$	$y_{82}^{(m)}$	$y_{21}^{(m)}$	$y_{51}^{(m)}$	$y_{81}^{(m)}$	$y_{10}^{(m)}$	$y_{40}^{(m)}$	$y_{70}^{(m)}$	$y_{31}^{(m)}$	$y_{61}^{(m)}$	$y_{91}^{(m)}$	$y_{32}^{(m)}$	$y_{62}^{(m)}$	$y_{92}^{(m)}$	$y_{33}^{(m)}$							7
$z_{83}^{(m)}$	$z_{22}^{(m)}$	$z_{52}^{(m)}$	$z_{82}^{(m)}$	$z_{21}^{(m)}$	$z_{51}^{(m)}$	$z_{81}^{(m)}$	$z_{10}^{(m)}$	$z_{40}^{(m)}$	$z_{70}^{(m)}$	$z_{31}^{(m)}$	$z_{61}^{(m)}$	$z_{91}^{(m)}$	$z_{32}^{(m)}$	$z_{62}^{(m)}$	$z_{92}^{(m)}$	$z_{33}^{(m)}$	$z_{63}^{(m)}$	$z_{93}^{(m)}$		$z_{64}^{(m)}$	$z_{94}^{(m)}$		7
				⋮				⋮					⋮					⋮					

$$v_n(\psi) = \sum_{m=1}^{\infty} V_m \sin m\psi \tag{34b}$$

$$w_n(\psi) = \sum_{m=0}^{\infty} W_m \cos m\psi \tag{34c}$$

corresponding to the eigenvalue λ are called the eigenfunctions or mode shapes. In order to effect a solution, we truncate the infinite set of equations [Eqs. (33)] and employ a matrix iteration technique [Ref. (26)] to determine, from Rayleigh's quotient, the lowest eigenvalue $\bar{\lambda}$ of the resulting finite equation system. Next, the size of the equation system is increased and the lowest eigenvalue of the new system is determined. This procedure is repeated until the successive values of $\bar{\lambda}$ have stabilized to the value λ_{cr}. Then by varying n, we obtain a set of these values of λ_{cr}, one for each n, and the solution to the buckling problem is given by the minimum value of λ_{cr} in this set. This procedure was programmed for the IBM 7094 digital computer and the numerical results are presented in the next chapter. At this point we may remark that, although a mathematical proof of convergence has not been presented here, the numerical results indicate convergence of the eigenvalues within a wide range of values of geometric parameters.

2.2.2 Mode B

For the buckling mode which is antimetric about the plane $\psi = 0, \pi$ (plane A-A in Fig. 1), the displacement components $u_n(\psi)$, $v_n(\psi)$, and $w_n(\psi)$ are represented by the Fourier series:

$$u_n(\psi) = \sum_{m=1}^{\infty} \tilde{U}_m \sin(m\psi) = \sum_{m=1}^{\infty} \tilde{U}_m S_m \tag{35a}$$

$$v_n(\psi) = \sum_{m=0}^{\infty} \tilde{V}_m \cos m\psi = \sum_{m=0}^{\infty} \tilde{V}_m C_m \tag{35b}$$

$$w_n(\psi) = \sum_{m=1}^{\infty} \tilde{W}_m \sin m\psi = \sum_{m=1}^{\infty} \tilde{W}_m S_m \quad . \tag{35c}$$

A detailed analysis of buckling in the mode which is symmetric about the plane $\psi = 0, \pi$ has been given in the preceding subsection. Corresponding equations for the antimetric mode can be obtained in the same way and therefore a detailed analysis for this buckling mode will not be presented here.

The stability equations for the buckling mode which is antimetric about the plane $\psi = 0, \pi$ are

$$\begin{aligned} &\tilde{z}_{10}^{(m)} \tilde{U}_m + \sum_{r=1}^{3} \tilde{z}_{2r}^{(m)} \tilde{U}_{|m-r|} + \sum_{r=1}^{3} \tilde{z}_{3r}^{(m)} \tilde{U}_{m+r} + \tilde{z}_{40}^{(m)} \tilde{V}_m + \sum_{r=1}^{4} \tilde{z}_{5r}^{(m)} \tilde{V}_{|m-r|} \\ &+ \sum_{r=1}^{4} \tilde{z}_{6r}^{(m)} \tilde{V}_{m+r} + \tilde{z}_{70}^{(m)} \tilde{W}_m + \sum_{r=1}^{4} \tilde{z}_{8r}^{(m)} \tilde{W}_{|m-r|} \\ &+ \sum_{r=1}^{4} \tilde{z}_{9r}^{(m)} \tilde{W}_{m+r} = 0 \quad , \quad m = (1, 2, \ldots .) \end{aligned} \tag{36a}$$

$$\begin{aligned} &\tilde{x}_{10}^{(m)} \tilde{U}_m + \tilde{x}_{21}^{(m)} \tilde{U}_{|m-1|} + \tilde{x}_{31}^{(m)} \tilde{U}_{m+1} + \tilde{x}_{40}^{(m)} \tilde{V}_m + \sum_{r=1}^{2} \tilde{x}_{5r}^{(m)} \tilde{V}_{|m-r|} \\ &+ \sum_{r=1}^{2} \tilde{x}_{6r}^{(m)} \tilde{V}_{m+r} + \tilde{x}_{70}^{(m)} \tilde{W}_m + \sum_{r=1}^{2} \tilde{x}_{8r}^{(m)} \tilde{W}_{|m-r|} \\ &+ \sum_{r=1}^{2} \tilde{x}_{9r}^{(m)} \tilde{W}_{m+r} = 0 \quad , \quad m = (0, 1, 2 \ldots) \end{aligned} \tag{36b}$$

$$\tilde{y}_{10}^{(m)} \tilde{U}_m + \sum_{r=1}^{3} \tilde{y}_{2r}^{(m)} \tilde{U}_{|m-r|} + \sum_{r=1}^{3} \tilde{y}_{3r}^{(m)} \tilde{U}_{m+r} + \tilde{y}_{40}^{(m)} \tilde{V}_m + \sum_{r=1}^{2} \tilde{y}_{5r}^{(m)} \tilde{V}_{|m-r|}$$

$$+ \sum_{r=1}^{2} \tilde{y}_{6r}^{(m)} \tilde{V}_{m+r} + \tilde{y}_{70}^{(m)} \tilde{W}_m + \sum_{r=1}^{2} \tilde{y}_{8r}^{(m)} \tilde{W}_{|m-r|}$$

$$+ \sum_{r=1}^{2} \tilde{y}_{9r}^{(m)} \tilde{W}_{m+r} = 0 , \quad m = (1,2,\ldots) \tag{36c}$$

where

$$\tilde{z}_{ij}^{(m)} = \hat{\zeta}_{ij}^{(m)} - \omega \bar{\zeta}_{ij}^{(m)} \tag{37a}$$

$$\tilde{x}_{ij}^{(m)} = \hat{\xi}_{ij}^{(m)} - \omega \bar{\xi}_{ij}^{(m)} \tag{37b}$$

$$\tilde{y}_{ij}^{(m)} = \hat{\eta}_{ij}^{(m)} - \omega \bar{\eta}_{ij}^{(m)} \tag{37c}$$

The coefficients on the right-hand sides of Eqs. (37) are given by

$$\bar{\xi}_{10}^{(m)} = -\left[2 h_{10} - 2m^2 h_{30}\right] \tag{38}$$

$$\bar{\xi}_{2r}^{(m)} = -\left[-\epsilon_{mr} h_{1r} + |m-r| h_{2r} + \epsilon_{mr} |m-r|^2 h_{3r}\right] \quad (r = 1,2,3)$$

$$\bar{\xi}_{3r}^{(m)} = -\left[h_{1r} - (m+r) h_{2r} - (m+r)^2 h_{3r}\right] \quad (r = 1,2,3)$$

$$\bar{\xi}_{40}^{(m)} = -\left[-2m\, h_{50} + 2m^3 h_{70}\right]$$

$$\bar{\xi}_{5r}^{(m)} = -\Big[(1+\delta_{mr}) h_{4r} + \epsilon_{mr} |m-r| h_{5r} - |m-r|^2 h_{6r}$$

$$- \epsilon_{mr} |m-r|^3 h_{7r}\Big] \quad (r = 1,2,3,4)$$

$$\bar{\xi}_{6r}^{(m)} = -\left[-h_{4r} - (m+r) h_{5r} + (m+r)^2 h_{6r} + (m+r)^3 h_{7r}\right] \quad (r = 1,2,3,4)$$

$$\bar{\xi}_{70}^{(m)} = -\left[2\, h_{80} - 2m^2\, h_{10,0} + 2m^4\, h_{12,0}\right] \qquad \text{(38 cont'd)}$$

$$\bar{\xi}_{8r}^{(m)} = -\Big[-\epsilon_{mr}\, h_{8r} + |m - r|\, h_{9r} + \epsilon_{mr}\, |m - r|^2\, h_{10,r} - |m - r|^3\, h_{11,r}$$

$$- \epsilon_{mr}\, |m - r|^4\, h_{12,r}\Big] \quad (r = 1,2,3,4)$$

$$\bar{\xi}_{9r}^{(m)} = -\Big[h_{8r} - (m + r)\, h_{9r} - (m + r)^2\, h_{10,r} + (m + r)^3\, h_{11,r}$$

$$+ (m + r)^4\, h_{12,r}\Big] \quad (r = 1,2,3,4)$$

$$\hat{\xi}_{10}^{(m)} = 2\, c_{10}$$

$$\hat{\xi}_{2r}^{(m)} = -\epsilon_{mr}\, c_{1r} \qquad (r = 1,2,3)$$

$$\hat{\xi}_{3r}^{(m)} = c_{1r} \qquad (r = 1,2,3)$$

$$\hat{\xi}_{40}^{(m)} = -2m\, c_{50}$$

$$\hat{\xi}_{5r}^{(m)} = (1 + \epsilon_{mr})\, c_{4r} + \epsilon_{mr}\, |m - r|\, c_{5r} \qquad (r = 1,2,3,4)$$

$$\hat{\xi}_{6r}^{(m)} = -c_{4r} - (m + r)\, c_{5r} \qquad (r = 1,2,3,4)$$

$$\hat{\xi}_{70}^{(m)} = 2\, c_{80} - 2m^2\, c_{10,0}$$

$$\hat{\xi}_{8r}^{(m)} = -\epsilon_{mr}\, c_{8r} + |m - r|\, c_{9r} + \epsilon_{mr}\, |m - r|^2\, c_{10,r} \qquad (r = 1,2,3,4)$$

$$\hat{\xi}_{9r}^{(m)} = c_{8r} - (m + r)\, c_{9r} - (m + r)^2\, c_{10r} \qquad (r = 1,2,3,4)$$

$$\bar{\xi}_{10}^{(m)} = -\left[2m\, f_{20}\right]$$

$$\bar{\xi}_{21}^{(m)} = -\left[\epsilon_{m1}\, f_{11} + |m - 1|\, f_{21}\right]$$

$$\bar{\xi}_{31}^{(m)} = -(1 - \delta_{mo})\left[f_{11} + (m + 1)\, f_{21}\right]$$

$$\bar{\xi}_{40}^{(m)} = -\left[2 f_{40} - 2m^2 f_{60}\right]$$ (38 cont'd)

$$\bar{\xi}_{5r}^{(m)} = -\left[(1 + \delta_{mr}) f_{4r} - \epsilon_{mr} |m - r| f_{5r} - |m - r|^2 f_{6r}\right] \quad (r = 1, 2)$$

$$\bar{\xi}_{6r}^{(m)} = -(1 - \delta_{mo}) \left[f_{4r} - (m + r) f_{5r} - (m + r)^2 f_{6r}\right] \quad (r = 1, 2)$$

$$\bar{\xi}_{70}^{(m)} = -\left[2m f_{90}\right]$$

$$\bar{\xi}_{8r}^{(m)} = -\left[\delta_{r1} \epsilon_{mr} f_{8r} + |m - r| f_{9r}\right] \quad (r = 1, 2)$$

$$\bar{\xi}_{9r}^{(m)} = -(1 - \delta_{mo}) \left[\delta_{r1} f_{8r} + (m + r) f_{9r}\right] \quad (r = 1, 2)$$

$$\hat{\xi}_{10}^{(m)} = 0$$

$$\hat{\xi}_{21}^{(m)} = 0$$

$$\hat{\xi}_{31}^{(m)} = 0$$

$$\hat{\xi}_{40}^{(m)} = 2 a_{40}$$

$$\hat{\xi}_{5r}^{(m)} = (1 + \delta_{mr}) a_{4r} \quad (r = 1, 2)$$

$$\hat{\xi}_{6r}^{(m)} = (1 - \delta_{mo}) a_{4r} \quad (r = 1, 2)$$

$$\hat{\xi}_{70}^{(m)} = 2m a_{90}$$

$$\hat{\xi}_{8r}^{(m)} = |m - r| a_{9r} \quad (r = 1, 2)$$

$$\hat{\xi}_{9r}^{(m)} = (1 - \delta_{mo}) (m + r) a_{9r} \quad (r = 1, 2)$$

$$\bar{\eta}_{10}^{(m)} = -\left[2 g_{10} - 2m^2 g_{30}\right]$$

$$\bar{\eta}_{2r}^{(m)} = -\left[- \epsilon_{mr} g_{1r} + |m - r| g_{2r} + \epsilon_{mr} |m - r|^2 g_{3r}\right] \quad (r = 1, 2, 3)$$

d) $$\bar{\eta}_{3r}^{(m)} = -\left[g_{1r} - (m + r)\, g_{2r} - (m + r)^2\, g_{3r}\right] \quad (r = 1,2,3)$$ (38 concl'd)

$$\bar{\eta}_{40}^{(m)} = -\left[-2m\, g_{50}\right]$$

$$\bar{\eta}_{5r}^{(m)} = -\left[(1 + \delta_{mr})\, g_{4r} + \epsilon_{mr}\, |m - r|\, g_{5r}\right] \quad (r = 1,2)$$

$$\bar{\eta}_{6r}^{(m)} = -\left[-g_{4r} - (m + r)\, g_{5r}\right] \quad (r = 1,2)$$

$$\bar{\eta}_{70}^{(m)} = -\left[2\, g_{80}\right]$$

$$\bar{\eta}_{8r}^{(m)} = -\left[-\epsilon_{mr}\, g_{8r}\right] \quad (r = 1,2)$$

$$\bar{\eta}_{9r}^{(m)} = -\left[g_{8r}\right] \quad (r = 1,2)$$

$$\hat{\eta}_{10}^{(m)} = 2\, b_{10} - 2m^2\, b_{30}$$

$$\hat{\eta}_{2r}^{(m)} = -\epsilon_{mr}\, b_{1r} + |m - r|\, b_{2r} + \epsilon_{mr}\, |m - r|^2\, b_{3r} \quad (r = 1,2,3)$$

$$\hat{\eta}_{3r}^{(m)} = b_{1r} - (m + r)\, b_{2r} - (m + r)^2\, b_{3r} \quad (r = 1,2,3)$$

$$\hat{\eta}_{40}^{(m)} = 0$$

$$\hat{\eta}_{5r}^{(m)} = (1 + \delta_{mr})\, b_{4r} \quad (r = 1,2)$$

$$\hat{\eta}_{6r}^{(m)} = -b_{4r} \quad (r = 1,2)$$

$$\hat{\eta}_{70}^{(m)} = 2\, b_{80}$$

$$\hat{\eta}_{8r}^{(m)} = -\epsilon_{mr}\, b_{8r} \quad (r = 1,2)$$

$$\hat{\eta}_{9r}^{(m)} = b_{8r} \quad (r = 1,2)$$

The coefficients a , b , c , f , g , and h in Eqs. (38) are given by Eqs. (15).

By letting m take on the values $m = 0, 1, 2, \ldots$ in Eqs. (36), we obtain an infinite system of linear homogeneous algebraic equations in which the unknowns are the Fourier coefficients $\tilde{U}_m$, $\tilde{V}_m$, and $\tilde{W}_m$. The coefficients in this system of equations are shown in Table 2. Using matrix notation, we rewrite Eqs. (36) as:

$$\left[\tilde{R}\right]\left\{\tilde{V}\right\} - \omega\left[\tilde{S}\right]\left\{\tilde{V}\right\} = \left\{0\right\} \tag{39}$$

where $\left[\tilde{R}\right]$ and $\left[\tilde{S}\right]$ are square matrices formed by the coefficients $\left(\hat{\xi}_{ij}^{(m)}, \hat{\eta}_{ij}^{(m)}, \hat{\zeta}_{ij}^{(m)}\right)$ and $\left(\bar{\xi}_{ij}^{(m)}, \bar{\eta}_{ij}^{(m)}, \bar{\zeta}_{ij}^{(m)}\right)$, respectively; $\left\{\tilde{V}\right\}$ is a column vector formed by the unknown Fourier coefficients $\tilde{U}_m$, $\tilde{V}_m$, and $\tilde{W}_m$.

The buckling load is obtained by the method discussed in the preceding subsection and numerical results are presented in the next chapter.

3. Stability Equations for a Sphere

The stability equations for axially symmetric buckling of a sphere subject to gas pressure are obtained by setting $u = n = \alpha = 0$ in the stability equations for a torus $\left[\text{Eqs. (10)}\right]$. As a result, Eq. (10c) which corresponds to equilibrium in the circumferential direction is identically satisfied. And, with the aid of the trigonometric identities given in Eqs. (13), the non-zero coefficients in Eqs. (10a and 10b) are:

$$h_4 = k\left[\frac{1}{4}(3-\nu)S_2 + \frac{1}{8}(1-\nu)S_4\right] - (1+\nu)\left[\frac{1}{4}S_2 + \frac{1}{8}S_4\right] = C_1^{\,4}\left\{k\left[(2-\nu)T_1 + T_1^{\,3}\right] - (1+\nu)T_1\right\} \tag{40a}$$

$$h_5 = k\left[\frac{1}{2} + \frac{3}{8}\nu + \frac{1}{2}(1+\nu)C_2 + \frac{\nu}{8}C_4\right] + (1+\nu)\left(\frac{3}{8} + \frac{1}{2}C_1^{\,2} + \frac{1}{8}C_4\right) = C_1^{\,4}\left\{k\left[(1+\nu) + T_1^{\,2}\right] + (1+\nu)\right\} \tag{40b}$$

		Coefficients of: V_o	U_1	V_1	W_1	U_2	V_2	W_2	U_3	V_3	W_3	U_4	V_4	W_4	U_5	V…	V_4
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	1…	12
1	m = 0	$\tilde{x}^{(0)}_{40}$	$\tilde{x}^{(0)}_{21}$	$\tilde{x}^{(0)}_{51}$	$\tilde{x}^{(0)}_{81}$		$\tilde{x}^{(0)}_{52}$	$\tilde{x}^{(0)}_{82}$									
2		$\tilde{x}^{(1)}_{51}$	$\tilde{x}^{(1)}_{10}$	$\tilde{x}^{(1)}_{40}+\tilde{x}^{(1)}_{52}$	$\tilde{x}^{(1)}_{70}+\tilde{x}^{(1)}_{82}$	$\tilde{x}^{(1)}_{31}$	$\tilde{x}^{(1)}_{61}$	$\tilde{x}^{(1)}_{91}$		$\tilde{x}^{(1)}_{62}$	$\tilde{x}^{(1)}_{92}$						
3	m = 1	$\tilde{y}^{(1)}_{51}$	$\tilde{y}^{(1)}_{10}+\tilde{y}^{(1)}_{22}$	$\tilde{y}^{(1)}_{40}+\tilde{y}^{(1)}_{52}$	$\tilde{y}^{(1)}_{70}+\tilde{y}^{(1)}_{82}$	$\tilde{y}^{(1)}_{23}+\tilde{y}^{(1)}_{31}$	$\tilde{y}^{(1)}_{61}$	$\tilde{y}^{(1)}_{91}$	$\tilde{y}^{(1)}_{32}$	$\tilde{y}^{(1)}_{62}$	$\tilde{y}^{(1)}_{92}$	$\tilde{y}^{(1)}_{33}$					
4		$\tilde{z}^{(1)}_{51}$	$\tilde{z}^{(1)}_{10}+\tilde{z}^{(1)}_{22}$	$\tilde{z}^{(1)}_{40}+\tilde{z}^{(1)}_{52}$	$\tilde{z}^{(1)}_{70}+\tilde{z}^{(1)}_{82}$	$\tilde{z}^{(1)}_{23}+\tilde{z}^{(1)}_{31}$	$\tilde{z}^{(1)}_{53}+\tilde{z}^{(1)}_{61}$	$\tilde{z}^{(1)}_{83}+\tilde{z}^{(1)}_{91}$	$\tilde{z}^{(1)}_{32}$	$\tilde{z}^{(1)}_{54}+\tilde{z}^{(1)}_{62}$	$\tilde{z}^{(1)}_{84}+\tilde{z}^{(1)}_{92}$	$\tilde{z}^{(1)}_{33}$	$\tilde{z}^{(1)}_{63}$	$\tilde{z}^{(1)}_{93}$		[illegible]	$\tilde{z}^{(1)}_{63}$
5		$\tilde{x}^{(2)}_{52}$	$\tilde{x}^{(2)}_{21}$	$\tilde{x}^{(2)}_{51}$	$\tilde{x}^{(2)}_{81}$	$\tilde{x}^{(2)}_{10}$	$\tilde{x}^{(2)}_{40}$	$\tilde{x}^{(2)}_{70}$	$\tilde{x}^{(2)}_{31}$	$\tilde{x}^{(2)}_{61}$	$\tilde{x}^{(2)}_{91}$		$\tilde{x}^{(2)}_{62}$	$\tilde{x}^{(2)}_{92}$			$\tilde{x}^{(2)}_{62}$
6	m = 2	$\tilde{y}^{(2)}_{52}$	$\tilde{y}^{(2)}_{21}+y^{(2)}_{23}$	$\tilde{y}^{(2)}_{51}$	$\tilde{y}^{(2)}_{81}$	$\tilde{y}^{(2)}_{10}$	$\tilde{y}^{(2)}_{40}$	$\tilde{y}^{(2)}_{70}$	$\tilde{y}^{(2)}_{31}$	$\tilde{y}^{(2)}_{61}$	$\tilde{y}^{(2)}_{91}$	$\tilde{y}^{(2)}_{32}$	$\tilde{y}^{(2)}_{62}$	$\tilde{y}^{(2)}_{92}$	$\tilde{y}^{(2)}_{33}$		$\tilde{y}^{(2)}_{62}$
7		$\tilde{z}^{(2)}_{52}$	$\tilde{z}^{(2)}_{21}+\tilde{z}^{(2)}_{23}$	$\tilde{z}^{(2)}_{51}+\tilde{z}^{(2)}_{53}$	$\tilde{z}^{(2)}_{81}+\tilde{z}^{(2)}_{83}$	$\tilde{z}^{(2)}_{10}$	$\tilde{z}^{(2)}_{40}+\tilde{z}^{(2)}_{54}$	$\tilde{z}^{(2)}_{70}+\tilde{z}^{(2)}_{84}$	$\tilde{z}^{(2)}_{31}$	$\tilde{z}^{(2)}_{61}$	$\tilde{z}^{(2)}_{91}$	$\tilde{z}^{(2)}_{32}$	$\tilde{z}^{(2)}_{62}$	$\tilde{z}^{(2)}_{92}$	$\tilde{z}^{(2)}_{33}$	[illegible]	$\tilde{z}^{(2)}_{62}$
8				$\tilde{x}^{(3)}_{52}$	$\tilde{x}^{(3)}_{82}$	$\tilde{x}^{(3)}_{21}$	$\tilde{x}^{(3)}_{51}$	$\tilde{x}^{(3)}_{81}$	$\tilde{x}^{(3)}_{10}$	$\tilde{x}^{(3)}_{40}$	$\tilde{x}^{(3)}_{70}$	$\tilde{x}^{(3)}_{31}$	$\tilde{x}^{(3)}_{61}$	$\tilde{x}^{(3)}_{91}$		[illegible]	$\tilde{x}^{(3)}_{61}$
9	m = 3		$\tilde{y}^{(3)}_{22}$	$\tilde{y}^{(3)}_{52}$	$\tilde{y}^{(3)}_{82}$	$\tilde{y}^{(3)}_{21}$	$\tilde{y}^{(3)}_{51}$	$\tilde{y}^{(3)}_{81}$	$\tilde{y}^{(3)}_{10}$	$\tilde{y}^{(3)}_{40}$	$\tilde{y}^{(3)}_{70}$	$\tilde{y}^{(3)}_{31}$	$\tilde{y}^{(3)}_{61}$	$\tilde{y}^{(3)}_{91}$	$\tilde{y}^{(3)}_{32}$	[illegible]	$\tilde{y}^{(3)}_{61}$
10		$\tilde{z}^{(3)}_{53}$	$\tilde{z}^{(3)}_{22}$	$\tilde{z}^{(3)}_{52}+\tilde{z}^{(3)}_{54}$	$\tilde{z}^{(3)}_{82}+\tilde{z}^{(3)}_{84}$	$\tilde{z}^{(3)}_{21}$	$\tilde{z}^{(3)}_{51}$	$\tilde{z}^{(3)}_{81}$	$\tilde{z}^{(3)}_{10}$	$\tilde{z}^{(3)}_{40}$	$\tilde{z}^{(3)}_{70}$	$\tilde{z}^{(3)}_{31}$	$\tilde{z}^{(3)}_{61}$	$\tilde{z}^{(3)}_{91}$	$\tilde{z}^{(3)}_{32}$	[illegible]	$\tilde{z}^{(3)}_{61}$
11							$\tilde{x}^{(4)}_{52}$	$\tilde{x}^{(4)}_{82}$	$\tilde{x}^{(4)}_{21}$	$\tilde{x}^{(4)}_{51}$	$\tilde{x}^{(4)}_{81}$	$\tilde{x}^{(4)}_{10}$	$\tilde{x}^{(4)}_{40}$	$\tilde{x}^{(4)}_{70}$	$\tilde{x}^{(4)}_{31}$	[illegible]	$\tilde{x}^{(4)}_{40}$
12	m = 4		$\tilde{y}^{(4)}_{23}$			$\tilde{y}^{(4)}_{22}$	$\tilde{y}^{(4)}_{52}$	$\tilde{y}^{(4)}_{82}$	$\tilde{y}^{(4)}_{21}$	$\tilde{y}^{(4)}_{51}$	$\tilde{y}^{(4)}_{81}$	$\tilde{y}^{(4)}_{10}$	$\tilde{y}^{(4)}_{40}$	$\tilde{y}^{(4)}_{70}$	$\tilde{y}^{(4)}_{31}$	[illegible]	$\tilde{y}^{(4)}_{40}$
13		$\tilde{z}^{(4)}_{54}$	$\tilde{z}^{(4)}_{23}$	$\tilde{z}^{(4)}_{53}$	$\tilde{z}^{(4)}_{83}$	$\tilde{z}^{(4)}_{22}$	$\tilde{z}^{(4)}_{52}$	$\tilde{z}^{(4)}_{82}$	$\tilde{z}^{(4)}_{21}$	[illegible]	$\tilde{z}^{(4)}_{81}$	$\tilde{z}^{(4)}_{10}$	$\tilde{z}^{(4)}_{40}$	$\tilde{z}^{(4)}_{70}$	$\tilde{z}^{(4)}_{31}$	[illegible]	$\tilde{z}^{(4)}_{40}$
14										$\tilde{x}^{(m)}_{52}$	$\tilde{x}^{(m)}_{82}$	$\tilde{x}^{(m)}_{21}$	$\tilde{x}^{(m)}_{51}$	$\tilde{x}^{(m)}_{81}$	$\tilde{x}^{(m)}_{10}$	[illegible]	$\tilde{x}^{(m)}_{51}$
15	m = 5					$\tilde{y}^{(m)}_{23}$			$\tilde{y}^{(m)}_{22}$	$\tilde{y}^{(m)}_{52}$	$\tilde{y}^{(m)}_{82}$	$\tilde{y}^{(m)}_{21}$	$\tilde{y}^{(m)}_{51}$	$\tilde{y}^{(m)}_{81}$	$\tilde{y}^{(m)}_{10}$	[illegible]	$\tilde{y}^{(m)}_{51}$
16				$\tilde{z}^{(m)}_{54}$	$\tilde{z}^{(m)}_{84}$	$\tilde{z}^{(m)}_{23}$	$\tilde{z}^{(m)}_{53}$	$\tilde{z}^{(m)}_{83}$	$\tilde{z}^{(m)}_{22}$	$\tilde{z}^{(m)}_{52}$	$\tilde{z}^{(m)}_{82}$	$\tilde{z}^{(m)}_{21}$	$\tilde{z}^{(m)}_{51}$	$\tilde{z}^{(m)}_{81}$	$\tilde{z}^{(m)}_{10}$	[illegible]	$\tilde{z}^{(m)}_{51}$
17													$\tilde{x}^{(m)}_{52}$	$\tilde{x}^{(m)}_{82}$	$\tilde{x}^{(m)}_{21}$	[illegible]	$\tilde{x}^{(m)}_{52}$
18	m = 6								$\tilde{y}^{(m)}_{23}$			$\tilde{y}^{(m)}_{22}$	$\tilde{y}^{(m)}_{52}$	$\tilde{y}^{(m)}_{82}$	$\tilde{y}^{(m)}_{21}$	[illegible]	$\tilde{y}^{(m)}_{52}$
19							$\tilde{z}^{(m)}_{54}$	$\tilde{z}^{(m)}_{84}$	$\tilde{z}^{(m)}_{23}$	$\tilde{z}^{(m)}_{53}$	$\tilde{z}^{(m)}_{83}$	$\tilde{z}^{(m)}_{22}$	$\tilde{z}^{(m)}_{52}$	$\tilde{z}^{(m)}_{82}$	$\tilde{z}^{(m)}_{21}$	[illegible]	$\tilde{z}^{(m)}_{52}$
20																[illegible]	
21	m = 7											$\tilde{y}^{(m)}_{23}$			$\tilde{y}^{(m)}_{22}$	[illegible]	
22										$\tilde{z}^{(m)}_{54}$	$\tilde{z}^{(m)}_{84}$	$\tilde{z}^{(m)}_{23}$	$\tilde{z}^{(m)}_{53}$	$\tilde{z}^{(m)}_{83}$	$\tilde{z}^{(m)}_{22}$	[illegible]	$\tilde{z}^{(m)}_{53}$
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THIS PATTERN REPEATS FOR m ≥ 5

Table 2
MODE B STABILITY EQUATIONS

V_4	W_4	U_5	V_5	W_5	U_6	V_6	W_6	U_7	V_7	W_7	U_8	V_8	W_8	U_9	V_9	W_9	U_{10}	V_{10}	W_{10}	U_{11}	V_{11}	W_{11}	...	
12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	...	
$\bar{z}^{(1)}_{63}$	$\bar{z}^{(1)}_{93}$		$\bar{z}^{(1)}_{64}$	$\bar{z}^{(1)}_{94}$																				
$\bar{x}^{(2)}_{62}$	$\bar{x}^{(2)}_{92}$																							
$\bar{y}^{(2)}_{62}$	$\bar{y}^{(2)}_{92}$	$\bar{y}^{(2)}_{33}$																						
$\bar{z}^{(2)}_{62}$	$\bar{z}^{(2)}_{92}$	$\bar{z}^{(2)}_{33}$	$\bar{z}^{(2)}_{63}$	$\bar{z}^{(2)}_{93}$		$\bar{z}^{(2)}_{64}$	$\bar{z}^{(2)}_{94}$																	
$\bar{x}^{(3)}_{61}$	$\bar{x}^{(3)}_{91}$		$\bar{x}^{(3)}_{62}$	$\bar{x}^{(3)}_{92}$																				
$\bar{y}^{(3)}_{61}$	$\bar{y}^{(3)}_{91}$	$\bar{y}^{(3)}_{32}$	$\bar{y}^{(3)}_{62}$	$\bar{y}^{(3)}_{92}$	$\bar{y}^{(3)}_{33}$																			
$\bar{z}^{(3)}_{61}$	$\bar{z}^{(3)}_{91}$	$\bar{z}^{(3)}_{32}$	$\bar{z}^{(3)}_{62}$	$\bar{z}^{(3)}_{92}$	$\bar{z}^{(3)}_{33}$	$\bar{z}^{(3)}_{63}$	$\bar{z}^{(3)}_{93}$		$\bar{z}^{(3)}_{64}$	$\bar{z}^{(3)}_{94}$														
$\bar{x}^{(4)}_{40}$	$\bar{x}^{(4)}_{70}$	$\bar{x}^{(4)}_{31}$	$\bar{x}^{(4)}_{61}$	$\bar{x}^{(4)}_{91}$		$\bar{x}^{(4)}_{62}$	$\bar{x}^{(4)}_{92}$																	
$\bar{y}^{(4)}_{40}$	$\bar{y}^{(4)}_{70}$	$\bar{y}^{(4)}_{31}$	$\bar{y}^{(4)}_{61}$	$\bar{y}^{(4)}_{91}$	$\bar{y}^{(4)}_{32}$	$\bar{y}^{(4)}_{62}$	$\bar{y}^{(4)}_{92}$	$\bar{y}^{(4)}_{33}$																
$\bar{z}^{(4)}_{40}$	$\bar{z}^{(4)}_{70}$	$\bar{z}^{(4)}_{31}$	$\bar{z}^{(4)}_{61}$	$\bar{z}^{(4)}_{91}$	$\bar{z}^{(4)}_{32}$	$\bar{z}^{(4)}_{62}$	$\bar{z}^{(4)}_{92}$	$\bar{z}^{(4)}_{33}$	$\bar{z}^{(4)}_{63}$	$\bar{z}^{(4)}_{93}$		$\bar{z}^{(4)}_{64}$	$\bar{z}^{(4)}_{94}$											
$\bar{x}^{(m)}_{51}$	$\bar{x}^{(m)}_{81}$	$\bar{x}^{(m)}_{10}$	$\bar{x}^{(m)}_{40}$	$\bar{x}^{(m)}_{70}$	$\bar{x}^{(m)}_{31}$	$\bar{x}^{(m)}_{61}$	$\bar{x}^{(m)}_{91}$		$\bar{x}^{(m)}_{62}$	$\bar{x}^{(m)}_{92}$														
$\bar{y}^{(m)}_{51}$	$\bar{y}^{(m)}_{81}$	$\bar{y}^{(m)}_{10}$	$\bar{y}^{(m)}_{40}$	$\bar{y}^{(m)}_{70}$	$\bar{y}^{(m)}_{31}$	$\bar{y}^{(m)}_{61}$	$\bar{y}^{(m)}_{91}$	$\bar{y}^{(m)}_{32}$	$\bar{y}^{(m)}_{62}$	$\bar{y}^{(m)}_{92}$	$\bar{y}^{(m)}_{33}$													5
$\bar{z}^{(m)}_{51}$	$\bar{z}^{(m)}_{81}$	$\bar{z}^{(m)}_{10}$	$\bar{z}^{(m)}_{40}$	$\bar{z}^{(m)}_{70}$	$\bar{z}^{(m)}_{31}$	$\bar{z}^{(m)}_{61}$	$\bar{z}^{(m)}_{91}$	$\bar{z}^{(m)}_{32}$	$\bar{z}^{(m)}_{62}$	$\bar{z}^{(m)}_{92}$	$\bar{z}^{(m)}_{33}$	$\bar{z}^{(m)}_{63}$	$\bar{z}^{(m)}_{93}$		$\bar{z}^{(m)}_{64}$	$\bar{z}^{(m)}_{94}$								
$\bar{x}^{(m)}_{52}$	$\bar{x}^{(m)}_{82}$	$\bar{x}^{(m)}_{21}$	$\bar{x}^{(m)}_{51}$	$\bar{x}^{(m)}_{81}$	$\bar{x}^{(m)}_{10}$	$\bar{x}^{(m)}_{40}$	$\bar{x}^{(m)}_{70}$	$\bar{x}^{(m)}_{31}$	$\bar{x}^{(m)}_{61}$	$\bar{x}^{(m)}_{91}$		$\bar{x}^{(m)}_{62}$	$\bar{x}^{(m)}_{92}$											
$\bar{y}^{(m)}_{52}$	$\bar{y}^{(m)}_{82}$	$\bar{y}^{(m)}_{21}$	$\bar{y}^{(m)}_{51}$	$\bar{y}^{(m)}_{81}$	$\bar{y}^{(m)}_{10}$	$\bar{y}^{(m)}_{40}$	$\bar{y}^{(m)}_{70}$	$\bar{y}^{(m)}_{31}$	$\bar{y}^{(m)}_{61}$	$\bar{y}^{(m)}_{91}$	$\bar{y}^{(m)}_{32}$	$\bar{y}^{(m)}_{62}$	$\bar{y}^{(m)}_{92}$	$\bar{y}^{(m)}_{33}$										6
$\bar{z}^{(m)}_{52}$	$\bar{z}^{(m)}_{82}$	$\bar{z}^{(m)}_{21}$	$\bar{z}^{(m)}_{51}$	$\bar{z}^{(m)}_{81}$	$\bar{z}^{(m)}_{10}$	$\bar{z}^{(m)}_{40}$	$\bar{z}^{(m)}_{70}$	$\bar{z}^{(m)}_{31}$	$\bar{z}^{(m)}_{61}$	$\bar{z}^{(m)}_{91}$	$\bar{z}^{(m)}_{32}$	$\bar{z}^{(m)}_{62}$	$\bar{z}^{(m)}_{92}$	$\bar{z}^{(m)}_{33}$	$\bar{z}^{(m)}_{63}$	$\bar{z}^{(m)}_{93}$		$\bar{z}^{(m)}_{64}$	$\bar{z}^{(m)}_{94}$					
			$\bar{x}^{(m)}_{52}$	$\bar{x}^{(m)}_{82}$	$\bar{x}^{(m)}_{21}$	$\bar{x}^{(m)}_{51}$	$\bar{x}^{(m)}_{81}$	$\bar{x}^{(m)}_{10}$	$\bar{x}^{(m)}_{40}$	$\bar{x}^{(m)}_{70}$	$\bar{x}^{(m)}_{31}$	$\bar{x}^{(m)}_{61}$	$\bar{x}^{(m)}_{91}$		$\bar{x}^{(m)}_{62}$	$\bar{x}^{(m)}_{92}$								
		$\bar{y}^{(m)}_{22}$	$\bar{y}^{(m)}_{52}$	$\bar{y}^{(m)}_{82}$	$\bar{y}^{(m)}_{21}$	$\bar{y}^{(m)}_{51}$	$\bar{y}^{(m)}_{81}$	$\bar{y}^{(m)}_{10}$	$\bar{y}^{(m)}_{40}$	$\bar{y}^{(m)}_{70}$	$\bar{y}^{(m)}_{31}$	$\bar{y}^{(m)}_{61}$	$\bar{y}^{(m)}_{91}$	$\bar{y}^{(m)}_{32}$	$\bar{y}^{(m)}_{62}$	$\bar{y}^{(m)}_{92}$	$\bar{y}^{(m)}_{33}$							7
$\bar{z}^{(m)}_{53}$	$\bar{z}^{(m)}_{83}$	$\bar{z}^{(m)}_{22}$	$\bar{z}^{(m)}_{52}$	$\bar{z}^{(m)}_{82}$	$\bar{z}^{(m)}_{21}$	$\bar{z}^{(m)}_{51}$	$\bar{z}^{(m)}_{81}$	$\bar{z}^{(m)}_{10}$	$\bar{z}^{(m)}_{40}$	$\bar{z}^{(m)}_{70}$	$\bar{z}^{(m)}_{31}$	$\bar{z}^{(m)}_{61}$	$\bar{z}^{(m)}_{91}$	$\bar{z}^{(m)}_{32}$	$\bar{z}^{(m)}_{62}$	$\bar{z}^{(m)}_{92}$	$\bar{z}^{(m)}_{33}$	$\bar{z}^{(m)}_{63}$	$\bar{z}^{(m)}_{93}$		$\bar{z}^{(m)}_{64}$	$\bar{z}^{(m)}_{94}$		
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.					.				.					.					.					
.					.				.					.					.					

$$h_6 = k\left|\frac{1}{2} S_2 + \frac{1}{4} S_4\right| = C_1^{\,4}\left|k\,(2\,T_1)\right|, \quad h_7 = k\left|-\frac{3}{8} - \frac{1}{2} C_2 - \frac{1}{8} C_4\right|$$

$$= C_1^{\,4}\left|-k\right| \tag{40c}$$

$$h_8 = (1+\nu)\left(\frac{3}{4} + C_2 + \frac{1}{4} C_4\right) = C_1^{\,4}\left|2\,(1+\nu)\right|, \quad h_9 = k\left|-\frac{1}{4}(3-\nu)\,S_2\right.$$

$$\left.- \frac{1}{8}(1-\nu)\,S_4\right| = C_1^{\,4}\left\{k\left|-(2-\nu)\,T_1 - T_1^{\,3}\right|\right\} \tag{40d}$$

$$h_{10} = k\left|-\left(\frac{1}{2} + \frac{3}{8}\nu\right) - \frac{1}{2}(1+\nu)\,C_2 - \frac{\nu}{8} C_4\right| = C_1^{\,4}\left|-(1+\nu) - T_1^{\,2}\right|,$$

$$h_{11} = k\left|-\frac{1}{2} S_2 - \frac{1}{4} S_4\right| = C_1^{\,4}\left|k(-2\,T_1)\right| \tag{40e}$$

$$h_{12} = k\left|\frac{3}{8} + \frac{1}{2} C_2 + \frac{1}{8} C_4\right| = C_1^{\,4}\left|k\right|, \quad c_4 = q\left|-\frac{1}{4} S_2 - \frac{1}{8} S_4\right| = C_1^{\,4}\left|q(-T_1)\right| \tag{40f}$$

$$c_5 = q\left|\frac{3}{8} + \frac{1}{2} C_2 + \frac{1}{8} C_4\right| = C_1^{\,4}\left|q\right|, \quad c_8 = q\left|\frac{3}{2} + 2\,C_1 + \frac{1}{2} C_4\right| = C_1^{\,4}\left|4q\right| \tag{40g}$$

$$c_9 = q\left|-\frac{1}{4} S_2 - \frac{1}{8} S_4\right| = C_1^{\,4}\left|q(-T_1)\right|, \quad c_{10} = q\left|\frac{3}{8} + \frac{1}{2} C_2 + \frac{1}{8} C_4\right| = C_1^{\,4}\left|q\right| \tag{40h}$$

$$f_4 = -\frac{1}{2}(1+\nu) + \frac{1}{2}(1-\nu)\,C_2 = C_1^{\,2}\left\{-\left(\nu + T_1^{\,2}\right)\right\}, \quad f_5 = -\frac{1}{2} S_2 = C_1^{\,2}\left|-T_1\right| \tag{40i}$$

$$f_6 = \frac{1}{2} + \frac{1}{2} C_2 = C_1^{\,2}\left|1\right|, \quad f_8 = (1+\nu)\left(\frac{1}{2} + \frac{1}{2} C_2\right) = C_1^{\,2}\left|1+\nu\right| \tag{40j}$$

$$a_4 = q\left|-\frac{1}{2} - \frac{1}{2} C_2\right| = C_1^{\,2}\left|-q\right|, \quad a_9 = q\left|\frac{1}{2} + \frac{1}{2} C_2\right| = C_1^{\,2}\left|q\right| \tag{40h}$$

where

$$q = \frac{1}{2}(1-\nu^2)\lambda \quad , \tag{41}$$

and

$$T_1 = \tan \psi \quad . \tag{42}$$

With a trivial change in the independent variable

$$\phi = \psi + \frac{\pi}{2} ,$$

we arrive at the following form of the stability equations for symmetric buckling of a sphere under hydrostatic pressure:

$$\begin{aligned}(1+\nu)\left[v \cot\phi + \dot{v} + 2w\right] + k\Big[&-v(2-\nu+\cot^2\phi)\cot\phi + \dot{v}(1+\nu+\cot^2\phi) \\ &- 2\cot\phi\,\ddot{v} - \dddot{v} + \dot{w}(2-\nu+\cot^2\phi)\cot\phi - \ddot{w}(1+\nu+\cot^2\phi) \\ &+ 2\cot\phi\,\dddot{w} + \ddddot{w}\Big] + q\left[v\cot\phi + \dot{v} + 4w + \dot{w}\cot\phi + \ddot{w}\right] = 0 \end{aligned} \tag{43a}$$

$$-(\nu+\cot^2\phi)v + \dot{v}\cot\phi + \ddot{v} + \dot{w}(1+\nu) - q(v-\dot{w}) = 0 . \tag{43b}$$

These equations are identical to Eqs. (VII-76a) and (VII-76b) of Ref. (13) provided that, as was done here, the transverse shear force Q_ϕ is omitted from the equation of equilibrium in the meridional direction [Eq. (VII-76a) of Ref. (13)].

VII

TOROIDAL SHELL UNDER EXTERNAL PRESSURE – NUMERICAL RESULTS

Numerical results for the buckling of a complete toroidal shell under uniform external pressure are presented in this chapter. All numerical results are for a value of Poisson's ratio, $\nu = 0.3$.

1. Numerical Results

The procedure used to obtain the lowest eigenvalue will now be illustrated by means of a typical example. Let us consider a toroidal shell whose geometric parameters are $a/h = 100$ and $b/a = 4$, and which buckles in Mode A with $n = 2$ circumferential waves. From these values, all of the coefficients $r_{k,\ell}$, $s_{k,\ell}$ in the infinite system of stability equations [Eqs. (VI – 33)] may be determined. A matrix iteration technique is then used to get the lowest eigenvalue $\bar{\lambda}$ of a finite system of equations which is obtained through truncation of the infinite system of equations. The size of the finite system of equations is determined from the number of harmonics M used in the series expansions for the displacement components $u_n(\psi)$, $v_n(\psi)$, and $w_n(\psi)$. That is, in Eqs. (VI – 20), the Fourier index m takes on the values $m = 0, 1, 2, \ldots, M$. The results obtained from the matrix iteration method are shown in Table 1 for a system of equations corresponding to $M = 14$. From Table 1, we see that four place accuracy in the eigenvalue $\bar{\lambda} = pa/Eh$ was achieved after eight iterations. Next, by assigning a sequence of values to M, we successively increase the size of the system of equations until no significant change occurs in the computed eigenvalue $\bar{\lambda}$. The results so

obtained are shown in Table 2. We see from Table 2 that the sequence of eigenvalues $\bar{\lambda}$ converged to the value $\lambda_{cr} = 0.1746 \times 10^{-3}$. By proceeding in this way, we can determine the eigenvalues λ_{cr} corresponding to different integer values of the number of circumferential waves n. Such results are given in Table 3 which shows that the lowest eigenvalue occurred at $n = 2$. To complete the analysis of the toroidal shell with $a/h = 100$ and $b/a = 4$, it is necessary to consider also buckling of the shell in Mode B and in the axially symmetric mode $(n = 0)$. The eigenvalues for these modes can be obtained by the same procedure used for Mode A.

Table 1

RESULTS OF MATRIX ITERATION

Mode A: $a/h = 100$, $b/a = 4$, $n = 2$, $M = 14$			
Iteration	$\frac{pa}{Eh} \times 10^3$	Iteration	$\frac{pa}{Eh} \times 10^3$
1	7.6551	9	0.1746
2	1.0454	10	0.1746
3	0.1973	11	0.1746
4	0.1763	12	0.1746
5	0.1752	13	0.1746
6	0.1747	14	0.1746
7	0.1746	15	0.1746
8	0.1746		

Table 2

EFFECT OF SIZE OF MATRIX ON COMPUTED BUCKLING PRESSURE

Mode A: a/h = 100, b/a = 4, n = 2			
M	$\frac{pa}{Eh} \times 10^3$	M	$\frac{pa}{Eh} \times 10^3$
1	5.0164	13	0.1746
2	1.1389	14	0.1746
3	0.4549	15	0.1746
4	0.2762	16	0.1746
5	0.2136	17	0.1746
6	0.1840	18	0.1746
7	0.1773	19	0.1746
8	0.1749	20	0.1746
9	0.1746	21	0.1746
10	0.1746	22	0.1746
11	0.1746	23	0.1746
12	0.1746	24	0.1746

Table 3

VARIATION OF BUCKLING PRESSURE WITH NUMBER OF CIRCUMFERENTIAL WAVES

Mode A: a/h = 100, b/a = 4			
n	$\frac{pa}{Eh} \times 10^3$	n	$\frac{pa}{Eh} \times 10^3$
1	0.6978	6	0.2878
2	0.1746	8	0.3873
3	0.1923	10	0.5167
4	0.2179	12	0.6714
5	0.2494	18	1.2427

The effect of the size of the system of equations on the computed eigenvalue is illustrated in Table 4 for other values of the geometric parameters a/h and b/a. Inspection of Table 4 reveals that the number of harmonics M required for the same degree of accuracy in the eigenvalue increases with increasing a/h and decreases with increasing b/a.

For all cases considered, the eigenvalues for the axially symmetric buckling mode were higher than the eigenvalues for the asymmetric buckling modes. For the same value of n, the eigenvalues corresponding to the two asymmetric buckling modes (i.e., Modes A and B) were always close to each other, and, as can be seen from Table 5, the eigenvalues for Mode A were sometimes higher and sometimes lower than the eigenvalues for Mode B. In all cases investigated, the lowest eigenvalue occurred at $n = 2$ for both asymmetric buckling modes. Some mode shapes for $n = 2$ are given in Figs. 1 and 2.

For the limiting case of a sphere $(b/a \rightarrow 0)$ under external pressure, the classical solution $p = 1.21\ Eh^2/a^2$ was here reproduced numerically for both the asymmetric and axially symmetric buckling modes. Results for the limiting case of axially symmetric buckling $(b/a \rightarrow \infty)$ can be compared to the critical load for an infinitely long cylinder under external pressure. However, in order to make the analyses comparable, we first have to modify the cylinder analysis through deletion of the transverse shear force from the in-surface equilibrium equation. Once this has been done, the results from the two analyses are identical.

Table 4

EFFECT OF SIZE OF MATRIX ON COMPUTED BUCKLING PRESSURE

Mode A, n = 2				
M	a/h=500, b/a=2	a/h=100, b/a=2	a/h=500, b/a=8	a/h=100, b/a=8
	$\frac{pa}{Eh} \times 10^4$	$\frac{pa}{Eh} \times 10^3$	$\frac{pa}{Eh} \times 10^4$	$\frac{pa}{Eh} \times 10^3$
6	3.9804	0.5176	0.3104	0.1157
7	2.3602	0.3886	0.2195	0.1134
8	1.5499	0.3203	0.1626	0.1132
9	1.0051	0.2949	0.1403	0.1132
10	0.7757	0.2848	0.1283	0.1132
11	0.5824	0.2824	0.1239	0.1132
12	0.4828	0.2813	0.1222	0.1132
13	0.4072	0.2812	0.1217	0.1132
14	0.3652	0.2811	0.1216	0.1132
15	0.3367	0.2811	0.1216	0.1132
16	0.3201	0.2811	0.1216	0.1132
17	0.3108	0.2811	0.1216	0.1132
18	0.3052	0.2811	0.1216	0.1132
19	0.3029	0.2811	0.1216	0.1132
20	0.3015	0.2811	0.1216	0.1132
21	0.3010	0.2811	0.1216	0.1132
22	0.3008	0.2811	0.1216	0.1132
23	0.3007	0.2811	0.1216	0.1132
24	0.3007	0.2811	0.1216	0.1132
⋮	⋮			
30	0.3007			

Table 5

COMPARISON OF RESULTS FOR
MODE A AND MODE B

(a) $a/h = 100$, $n = 2$		
b/a	$\frac{pa}{Eh} \times 10^3$	
	Mode A	Mode B
1.2	0.520	0.516
2	0.281	0.281
4	0.175	0.176
8	0.113	0.115
20	0.066	0.067

(b) $a/h = 500$, $n = 2$		
b/a	$\frac{pa}{Eh} \times 10^4$	
	Mode A	Mode B
1.2	0.441	0.440
2	0.301	0.301
4	0.191	0.192
8	0.121	0.122
20	0.068	0.069

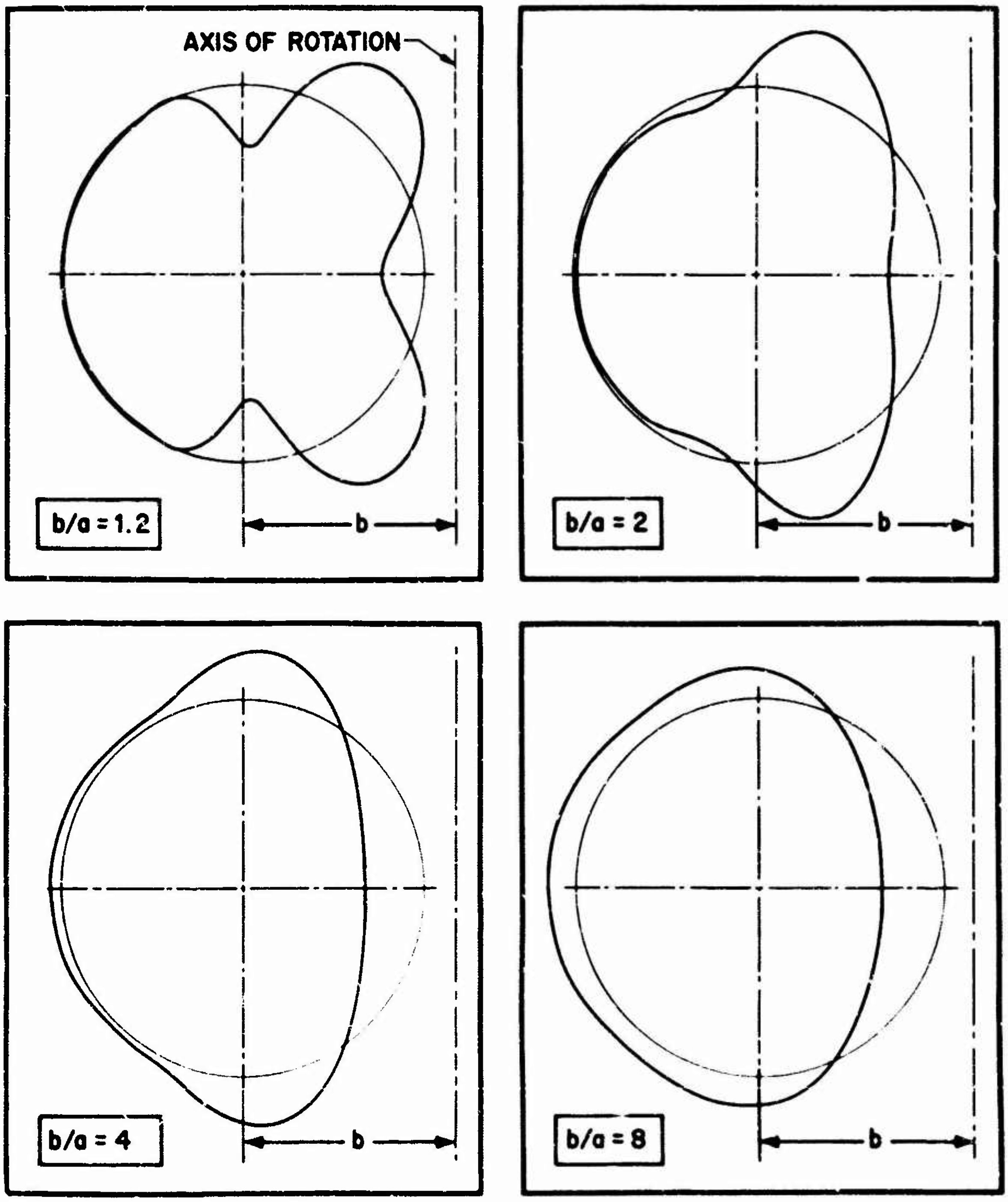

Fig. 1 Mode Shapes, a/h = 100, n = 2, Mode A

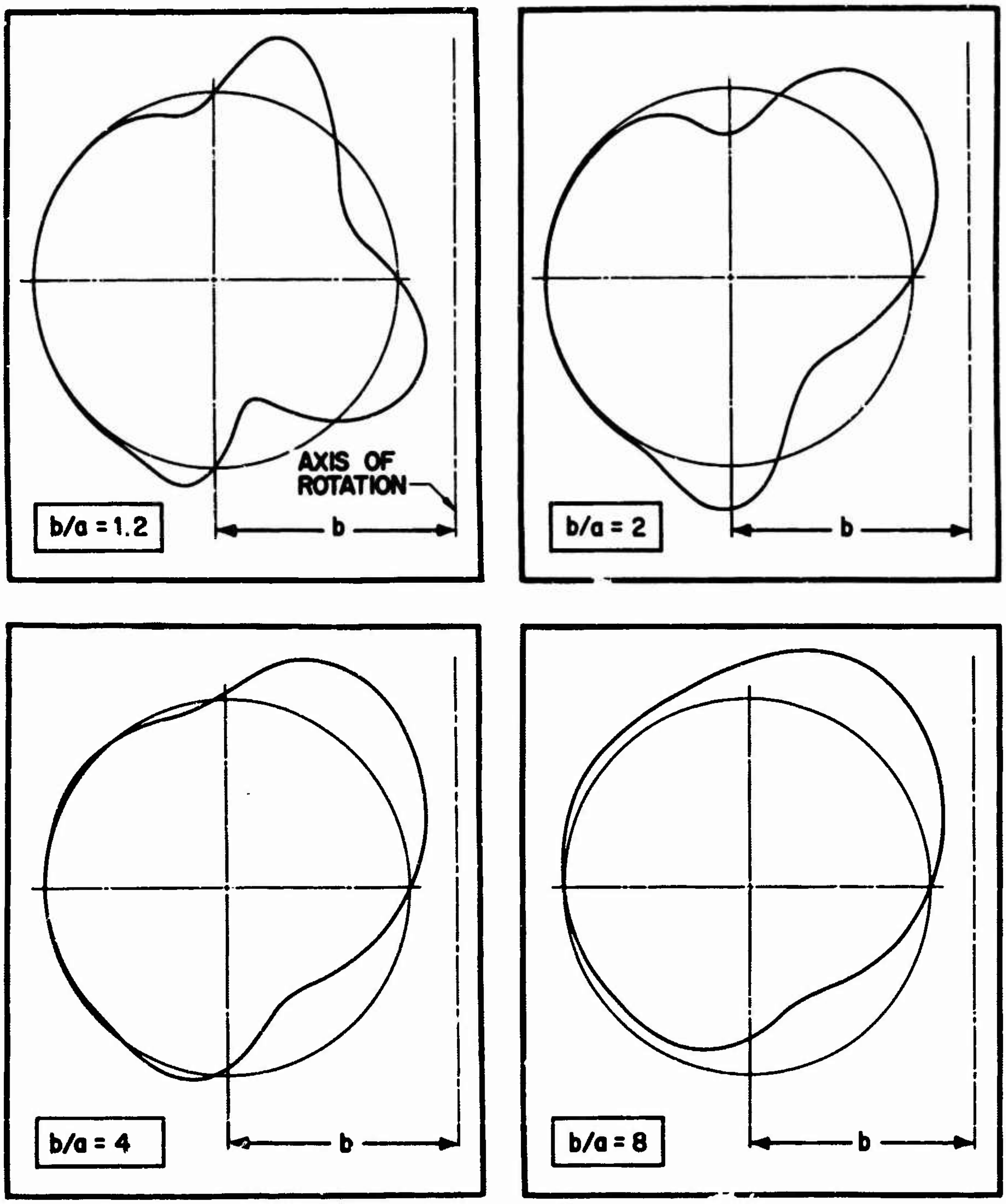

Fig. 2 Mode Shapes, $a/h = 100$, $n = 2$, Mode B

1.1 Buckling Curves

The accuracy of the numerical results, of course, depends on the use of a sufficient number of terms in the Fourier series expansions for the displacement components. On the other hand, the computer time increases rapidly with an increasing number of terms. The convergence of the method, therefore, was explored through calculation of buckling loads for fixed shell parameters and a successively increasing number of terms. By use of these exploratory calculations it was possible to establish, as a function of the geometrical parameters, the number of terms needed for 1%, or better, accuracy in the final results.

The computed critical values of the external pressures are shown in Fig. 3. From Fig. 3 we see, as expected, that the critical pressure p decreases with increasing a/h and increases with increasing a/b.

1.2 Rigid Body Modes

Let us denote the coefficients of the unknowns U_o, V_o, U_1, V_1, and W_1 in the i^{th} row of $[S]$ or $[\tilde{S}]$ by a_i, b_i, c_i, d_i, and e_i, respectively. Then, from Eqs. (VI – 15, 29, 30, and 32) or Eqs. (VI – 15, 36 through 38), we can obtain, for each value of i, the following relations:

(i) Mode A, n = 0

$$a_i = -\frac{1}{\alpha} c_i \tag{1}$$

(ii) Mode A, n = 1

$$a_i = e_i - d_i \tag{2}$$

(iii) Mode B, n = 0

$$d_i = -e_i \tag{3}$$

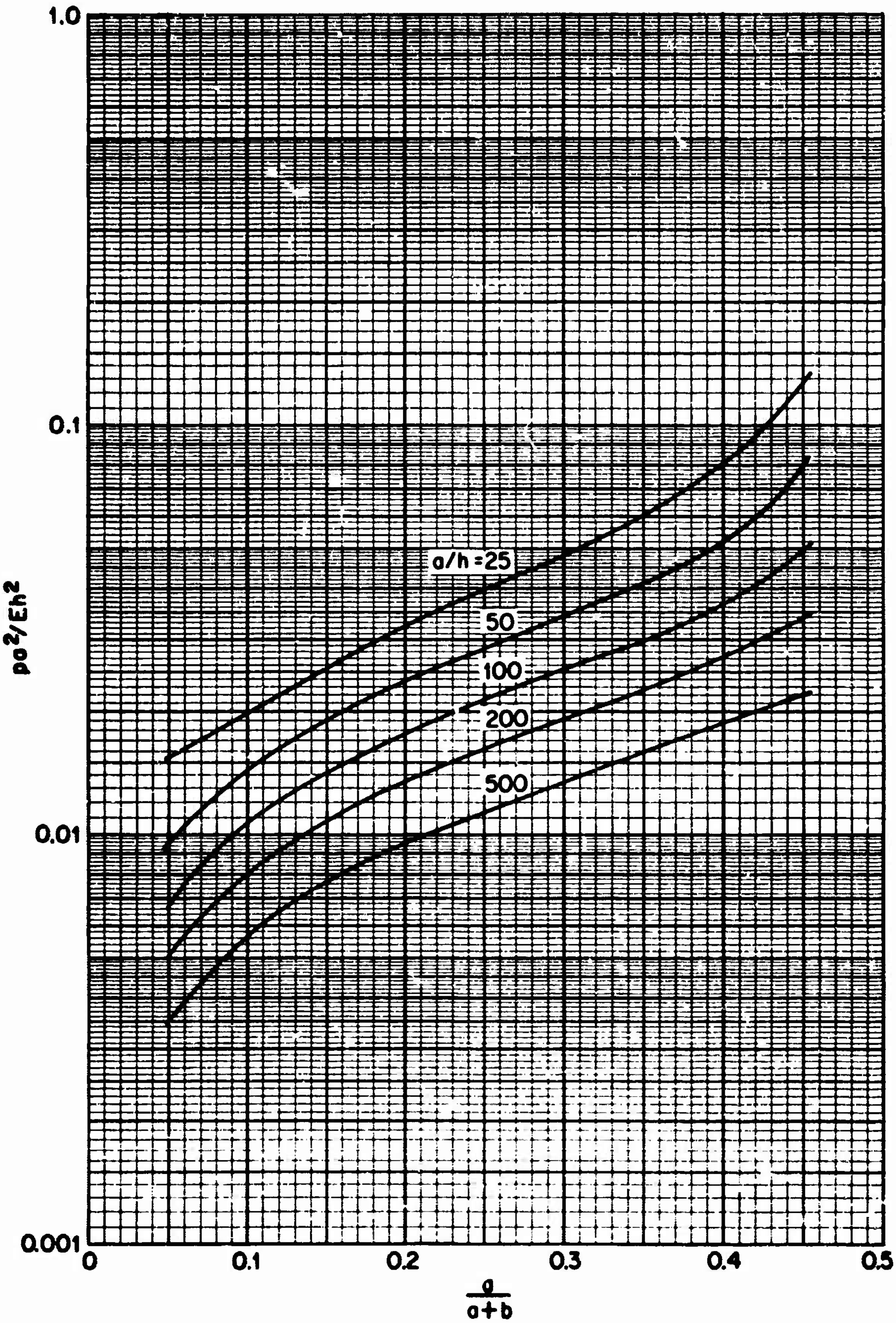

Fig. 3 Buckling Coefficients for Toroidal Shells Under Uniform External Pressure

(iv) Mode B, n = 1

$$b_i = -\alpha(d_i + e_i) - c_i . \tag{4}$$

With these relations, we can show that the equations which govern the stability of a toroidal shell [Eqs. (VI – 29) for Mode A or Eqs. (VI – 36) for Mode B] admit the following nontrivial solutions:

(i) Mode A, n = 0

$$\lambda = 0 \tag{5a}$$

$$u = r = a(\alpha + \cos\psi) \tag{5b}$$

$$v = 0 \tag{5c}$$

$$w = 0 \tag{5d}$$

(ii) Mode A, n = 1

$$\lambda = 0 \tag{6a}$$

$$u = \sin\theta \tag{6b}$$

$$v = \sin\psi\cos\theta \tag{6c}$$

$$w = -\cos\psi\cos\theta \tag{6d}$$

(iii) Mode B, n = 0

$$\lambda = 0 \tag{7a}$$

$$u = 0 \tag{7b}$$

$$v = \cos\psi \tag{7c}$$

$$w = \sin\psi \tag{7d}$$

(iv) Mode B, n = 1

$$\lambda = 0 \tag{8a}$$

$$u = \sin\psi\sin\theta \tag{8b}$$

$$v = (1 + \alpha\cos\psi)\cos\theta \tag{8c}$$

$$w = \alpha\sin\psi\cos\theta . \tag{8d}$$

Now these solutions, which occur at a zero value of the eigenvalue λ, are recognized as being the following rigid body modes for a toroidal shell:

Case (i) Rotation about the axis of revolution (Mode A, n = 0)

Case (ii) Translation in a plane normal to the axis of revolution (Mode A, n = 1).

Case (iii) Translation along the axis of revolution (Mode B, n = 0)

Case (iv) Rotation about an axis normal to the axis of revolution (Mode B, n = 1)

[Two more rigid body motions are obtained from Eqs. (6 and 8) through replacement of θ by $\theta + \pi/2$]

Since the proposal here is to obtain buckling loads, it is necessary in each of these cases to determine the lowest non-zero value of λ for which the stability equations admit a nontrivial solution and, of course, this value of λ was used to arrive at the numerical results already presented.

2. Comparison With Test Results

A comparison of the results of the present theory for complete toroidal shells and available results from tests conducted at Lockheed Missiles & Space Company is given in Table 6a. From Table 6a, we see that test and theory agreed to within 10%.

We note that the present results for complete toroidal shells can be used to predict buckling pressures for partial toroidal shells which are simply supported on the meridional edges $\theta = 0$, $\theta = \beta$. For example, the solution for a simply supported shell with $\beta = \pi$, $2\pi/3$, $\pi/2$, ... can be obtained

Table 6

COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

(a) Complete Toroidal Shells

a (in.)	b (in.)	h		$\frac{b}{a}$	$\frac{a}{h_{avg}}$	Material (Steel)	pa^2/Eh^2		$\frac{p_{theory} - p_{test}}{p_{theory}} \times 100$
		Nominal (in.)	Average (in.)				Test	Theory	
2.825	22.70	0.036	0.0395	8.04	71.5	17-7 PH	0.01182	0.01302	9.2
2.825	22.70	0.036	0.0383	8.04	73.8	17-7 PH	0.01370	0.01285	-6.6
2.825	22.70	0.036	0.0353	8.04	80.0	AM-350	0.01369	0.01240	-10.4

(b) 180° Toroidal Shell

a (in.)	b (in.)	h		$\frac{b}{a}$	$\frac{a}{h}$	Material (Titanium)	pa^2/Eh^2		$\frac{p_{theory} - p_{test}}{p_{theory}} \times 100$
		Nominal (in.)	Average (in.)				Test	Theory	
3.50	22.125	0.050	—	6.32	70	6Al-4V	0.0164	0.0152	-7.9

from the solution of a complete toroidal shell ~~which~~ buckles in $n = 2, 3, 4, \ldots$ circumferential waves. A 180° toroidal shell was tested at LMSC. A comparison between test and theory is shown in Table 6b. Again, the agreement was within 10%.

The infinitely long cylinder $(b/a = \infty)$ and the sphere $(b/a = 0)$ represent limiting cases of a toroidal shell. For external pressure loading, it is well known that the correlation between theory and experimental results is reasonably good for the cylinder and quite poor for the sphere. Now the correlation between the present theoretical results and the few available test results shown in Table 6a was quite good. However, the test results were for a slender torus $(b/a = 8)$, and it should be pointed out that the correlation might not be as satisfactory for smaller values of b/a.

3. Comparison with Previous Investigation

The only previous investigation of the stability of a toroidal shell under external pressure was performed by Machnig (Refs. 9 and 10). In the first of his papers (Ref. 9), Machnig studied both axially symmetric and asymmetric buckling modes and concluded, contrary to the present results, that the former buckling mode gives the smallest critical pressure. In his more recent paper (Ref. 10), Machnig considers only the axially symmetric mode. A comparison between Machnig's results and the results of the present analysis is given in Fig. 4.

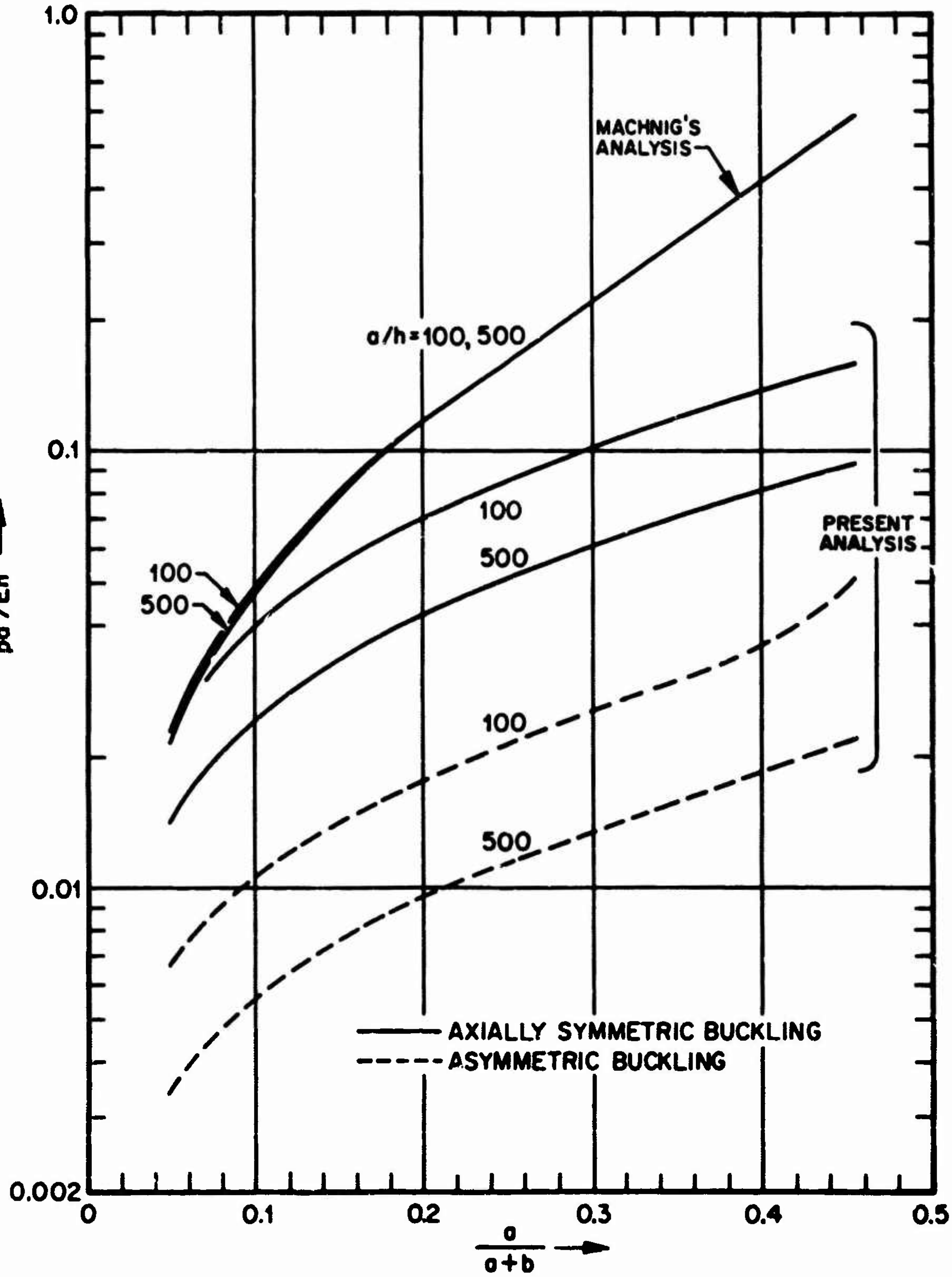

Fig. 4 Comparison With Previous Investigation

VIII

FREE VIBRATIONS OF PRESTRESSED SHELLS OF REVOLUTION

The components of the applied loads per unit area of the shell's middle surface are denoted by $p_\phi(\theta,\phi)$, $p_\theta(\theta,\phi)$, and $p_z(\theta,\phi)$ as shown in Fig. IV-1. The shell is in a state of equilibrium, called the prestressed state of equilibrium, under the action of the applied loads and the resulting membrane stress resultants $N_{\phi 0}(\theta,\phi)$, $N_{\theta 0}(\theta,\phi)$, $N_{\theta\phi 0}(\theta,\phi)$, and $N_{\phi\theta 0}(\theta,\phi)$. The effects of deformation of the prestressed shell will be neglected. The additional quantities that develop as the shell vibrates about its prestressed state of equilibrium are denoted by (see Figs. III-2 and 3) $N_\phi(\theta,\phi,t), \ldots, Q_\theta(\theta,\phi,t), \ldots, M_{\phi\theta}(\theta,\phi,t), \ldots, w(\theta,\phi,t)$. These additional or incremental quantities are considered to be infinitesimal. The mass per unit area of the shell's middle surface is denoted by μ.

1. Basic Equations for Shells of Revolution

The equations of motion for a differential element of the vibrating shell are obtained by addition of the inertia terms $-rr_1\mu\partial^2 v/\partial t^2$, $-rr_1\mu\partial^2 u/\partial t^2$, and $-rr_1\mu\partial^2 w/\partial t^2$ to the left-hand sides of the equations of force equilibrium derived for the stability problem of a shell of revolution [Eqs. (IV – 5)]:

$$\sum F_1 = (rN_\phi)^{\cdot} + r_1 N'_{\theta\phi} - r_1 N_\theta \cos\phi + \{-rQ_\phi - rN_{\phi 0}\omega_\theta - rN_{\phi\theta 0}\omega_\phi\}$$

$$+ [-r_1(N_{\theta 0}\omega_{z1})' - (rN_{\phi\theta 0}\omega_{z1})^{\cdot} - r_1 N_{\theta\phi 0}\omega_{z2}\cos\phi]$$

$$+ \delta_{ph} rr_1 p_z \omega_\theta - rr_1\mu \frac{\partial^2 v}{\partial t^2} = 0 \qquad (1a)$$

$$\sum F_2 = (rN_{\phi\theta})^{\cdot} + r_1 N'_\theta + r_1 N_{\theta\phi} \cos\phi + \{-r_1 Q_\theta \sin\phi - r_1 N_{\theta\phi 0}\omega_\theta \sin\phi$$

$$- r_1 N_{\theta 0}\omega_\phi \sin\phi\} + [\,-r_1 N_{\theta 0}\omega_{z1} \cos\phi + (rN_{\phi 0}\omega_{z2})^{\cdot} + r_1(N_{\theta\phi 0}\omega_{z2})'\,]$$

$$+ \delta_{ph} r r_1 p_z \omega_\phi - r r_1 \mu \frac{\partial^2 u}{\partial t^2} = 0 \tag{1b}$$

$$\sum F_3 = -r_1 N_\theta \sin\phi - rN_\phi - r_1 Q'_\theta - (rQ_\phi)^{\cdot} - (rN_{\phi 0}\omega_{\theta})^{\cdot} - r_1(N_{\theta\phi 0}\omega_\theta)'$$

$$- r_1(N_{\theta 0}\omega_\phi)' - (rN_{\phi\theta 0}\omega_\phi)^{\cdot} + [rN_{\phi\theta 0}\omega_{z1} - r_1 N_{\theta\phi 0}\omega_{z2} \sin\phi]$$

$$+ \delta_{ph} r r_1 p_z(\bar{\epsilon}_\phi + \bar{\epsilon}_\theta) - r r_1 \mu \frac{\partial^2 w}{\partial t^2} = 0 \tag{1c}$$

$$\sum M_1 = (rM_{\phi\theta})^{\cdot} + r_1 M'_\theta + r_1 M_{\theta\phi} \cos\phi - r r_1 Q_\theta = 0 \tag{1d}$$

$$\sum M_2 = -(rM_\phi)^{\cdot} - r_1 M'_{\theta\phi} + r_1 M_\theta \cos\phi + r r_1 Q_\phi = 0 \tag{1e}$$

The rotations ω_ϕ, ω_θ, ω_{z1}, and ω_{z2} are defined in Eqs. (III – 8) through Eq. (III – 10) and δ_{ph} is defined in Eq. (IV – 4).

The elastic laws which relate the incremental stress resultants $N_\phi, \ldots,$ $Q_\theta, \ldots, M_{\phi\theta}$ to the incremental displacement components are given by Eqs. (III – 18).

2. Shells of Revolution Under Axially Symmetric Loads

From Eqs. (1), we see that the vibrations of a shell of revolution are governed by a system of partial differential equations with variable coefficients. For the case of axially symmetric loading the coefficients in these equations are independent of the circumferential coordinate θ and the time t. Consequently, it is possible to separate the space and time variables and thus replace the system of partial differential equations by a system of ordinary

differential equations. Such a separation of variables is effected by means of the following Fourier series representation for the incremental displacement components:

$$u(\theta, \phi, t) = \sum_{n=1}^{\infty} u_n(\phi) \sin(n\theta)\, e^{i\omega_n t} \tag{2a}$$

$$v(\theta, \phi, t) = \sum_{n=0}^{\infty} v_n(\phi) \cos(n\theta)\, e^{i\omega_n t} \tag{2b}$$

$$w(\theta, \phi, t) = \sum_{n=0}^{\infty} w_n(\phi) \cos(n\theta)\, e^{i\omega_n t} \tag{2c}$$

Then by proceeding in the same way as in Chapter V, we obtain a set of three ordinary differential equations for the three displacement components $u_n(\phi)$, $v_n(\phi)$, and $w_n(\phi)$. For a given value of n, these equations are

$$\begin{aligned}&(h_1 + c_1)u_n + (h_2 + c_2)\dot{u}_n + (h_3 + c_3)\ddot{u}_n + (h_4 + c_4)v_n + (h_5 + c_5)\dot{v}_n \\ &+ (h_6 + c_6)\ddot{v}_n + (h_7 + c_7)\dddot{v}_n + (h_8 + c_8 + \tilde{c}_8)w_n + (h_9 + c_9)\dot{w}_n \\ &+ (h_{10} + c_{10})\ddot{w}_n + (h_{11} + c_{11})\dddot{w}_n + (h_{12} + c_{12})\ddddot{w}_n = 0\end{aligned} \tag{3a}$$

$$\begin{aligned}&(f_1 + a_1)u_n + (f_2 + a_2)\dot{u}_n + (f_3 + a_3)\ddot{u}_n + (f_4 + a_4 + \tilde{a}_4)v_n + (f_5 + a_5)\dot{v}_n \\ &+ (f_6 + a_6)\ddot{v}_n + (f_7 + a_7)\dddot{v}_n + (f_8 + a_8)w_n + (f_9 + a_9)\dot{w}_n + (f_{10} + a_{10})\ddot{w}_n \\ &+ (f_{11} + a_{11})\dddot{w}_n + (f_{12} + a_{12})\ddddot{w}_n = 0\end{aligned} \tag{3b}$$

$$(g_1 + b_1 + \tilde{b}_1)u_n + (g_2 + b_2)\dot{u}_n + (g_3 + b_3)\ddot{u}_n + (g_4 + b_4)v_n + (g_5 + b_5)\dot{v}_n$$
$$+ (g_6 + b_6)\ddot{v}_n + (g_7 + b_7)\dddot{v}_n + (g_8 + b_8)w_n + (g_9 + b_9)\dot{w}_n + (g_{10} + b_{10})\ddot{w}_n$$
$$+ (g_{11} + b_{11})\dddot{w}_n + (g_{12} + b_{12})\ddddot{w}_n = 0 \tag{3c}$$

where

$$\tilde{a}_4 = \tilde{b}_1 = -\tilde{c}_8 = +rr_1\mu\omega^2 \tag{4}$$

and all other coefficients in Eqs. (3) are given by Eqs. (V − 9). For convenience, a subscript n was deleted from a, . . . , h , $\tilde{a}_4$, $\tilde{b}_1$, $\tilde{c}_8$, and ω in Eqs. (3) and (4). We note that the functions $a(\phi)$, $b(\phi)$, and $c(\phi)$ in Eqs. (3) depend on the prestress quantities $N_{\phi 0}$, $N_{\theta 0}$, and p_z , whereas the coefficients in Eq. (4) depend on the frequency ω .

3. Free Vibrations of a Prestressed Toroidal Shell

The equations governing the free vibrations of a prestressed toroidal shell are obtained through specialization of the equations for a general shell of revolution [Eqs. (3)]. We consider the case in which the prestress is due to a uniform pressure p ; p is positive for external pressure and negative for internal pressure. The notation for a toroidal shell with a circular meridian is shown in Fig. VI-1.

It can be shown that a toroidal shell under an initial uniform pressure can vibrate in a mode which is either symmetric or antimetric about the plane $\psi = 0 , \pi$ (Plane A-A in Fig. VI-1). The symmetric mode is called Mode A, and the antimetric mode is called Mode B. These two modes are considered separately in the next two subsections.

3.1 Mode A

For the vibration mode which is symmetric about the plane $\psi = 0, \pi$, we let

$$u_n(\psi) = \sum_{m=0}^{\infty} U_m \cos m\psi = \sum_{m=0}^{\infty} U_m C_m \tag{5a}$$

$$v_n(\psi) = \sum_{m=1}^{\infty} V_m \sin m\psi = \sum_{m=1}^{\infty} V_m S_m \tag{5b}$$

$$w_n(\psi) = \sum_{m=0}^{\infty} W_m \cos m\psi = \sum_{m=0}^{\infty} W_m C_m \tag{5c}$$

where, for brevity, we have used the notations given by Eqs. (VI − 3). Then by inserting Eqs. (5) into Eqs. (3) and proceeding in the same way as in Chapter VI, we obtain the following form of the vibration equations:

$$\begin{aligned} & z_{10}^{(m)} U_m + \sum_{r=1}^{3} z_{2r}^{(m)} U_{|m-r|} + \sum_{r=1}^{3} z_{3r}^{(m)} U_{m+r} \\ & + z_{40}^{(m)} V_m + \sum_{r=1}^{4} z_{5r}^{(m)} V_{|m-r|} + \sum_{r=1}^{4} z_{6r}^{(m)} V_{m+r} \\ & + z_{70}^{(m)} W_m + \sum_{r=1}^{4} z_{8r}^{(m)} W_{|m-r|} + \sum_{r=1}^{4} z_{9r}^{(m)} W_{m+r} = 0 , \quad m = (0, 1, 2, \ldots) \end{aligned} \tag{6a}$$

$$\begin{aligned} & x_{10}^{(m)} U_m + x_{21}^{(m)} U_{|m-1|} + x_{31}^{(m)} U_{m+1} + x_{40}^{(m)} V_m + \sum_{r=1}^{2} x_{5r}^{(m)} V_{|m-r|} + \sum_{r=1}^{2} x_{6r}^{(m)} V_{m+r} \\ & + x_{70}^{(m)} W_m + \sum_{r=1}^{2} x_{8r}^{(m)} W_{|m-r|} + \sum_{r=1}^{2} x_{9r}^{(m)} W_{m+r} = 0 , \quad m = (1, 2, \ldots) \end{aligned} \tag{6b}$$

$$y_{10}^{(m)}U_m + \sum_{r=1}^{3} y_{2r}^{(m)}U_{|m-r|} + \sum_{r=1}^{3} y_{3r}^{(m)}U_{m+r} + y_{40}^{(m)}V_m + \sum_{r=1}^{2} y_{5r}^{(m)}V_{|m-r|}$$

$$+ \sum_{r=1}^{2} y_{6r}^{(m)}V_{m+r} + y_{70}^{(m)}W_m + \sum_{r=1}^{2} y_{8r}^{(m)}W_{|m-r|} + \sum_{r=1}^{2} y_{9r}^{(m)}W_{m+r} = 0$$

$$m = (0, 1, 2, \ldots) \qquad (6c)$$

where

$$z_{ij}^{(m)} = \overset{*}{z}{}_{ij}^{(m)} - \frac{1}{\Omega}\left[\bar{z}_{ij}^{(m)} - \lambda\hat{z}_{ij}^{(m)}\right] \qquad (7a)$$

$$x_{ij}^{(m)} = \overset{*}{x}{}_{ij}^{(m)} - \frac{1}{\Omega}\left[\bar{x}_{ij}^{(m)} - \lambda\hat{x}_{ij}^{(m)}\right] \qquad (7b)$$

$$y_{ij}^{(m)} = \overset{*}{y}{}_{ij}^{(m)} - \frac{1}{\Omega}\left[\bar{y}_{ij}^{(m)} - \lambda\hat{y}_{ij}^{(m)}\right] \qquad (7c)$$

and

$$\lambda = \frac{pa}{Eh} \qquad (8)$$

and

$$\Omega = \frac{\mu a^2 \omega^2}{Eh} \qquad (9)$$

The coefficients $\bar{z}_{ij}^{(m)}$, $\bar{x}_{ij}^{(m)}$, $\bar{y}_{ij}^{(m)}$, $\hat{z}_{ij}^{(m)}$, $\hat{x}_{ij}^{(m)}$, and $\hat{y}_{ij}^{(m)}$ in Eqs. (7) are given by Eqs. (VI – 32). The remaining nonzero coefficients in Eqs. (7) are

$$\overset{*}{z}{}_{70}^{(m)} = 2\tilde{c}_{80} \qquad (10a)$$

$$\overset{*}{z}{}_{8r}^{(m)} = (1 + \delta_{mr})\tilde{c}_{8r} \quad , \quad (r = 1, 2, 3, 4) \qquad (10b)$$

$$\overset{*}{z}{}_{9r}^{(m)} = (1 - \delta_{m0})\tilde{c}_{8r} \quad , \quad (r = 1, 2, 3, 4) \qquad (10c)$$

$$\overset{*}{x}{}_{40}^{(m)} = 2\tilde{a}_{40} \qquad (10d)$$

$$\overset{*}{x}{}^{(m)}_{5r} = -\epsilon_{mr}\tilde{a}_{4r} \quad , \quad (r = 1, 2) \tag{10e}$$

$$\overset{*}{x}{}^{(m)}_{6r} = \tilde{a}_{4r} \quad , \quad (r = 1, 2) \tag{10f}$$

$$\overset{*}{y}{}^{(m)}_{10} = 2\tilde{b}_{10} \tag{10g}$$

$$\overset{*}{y}{}^{(m)}_{2r} = (1 + \delta_{mr})\tilde{b}_{1r} \quad , \quad (r = 1, 2, 3) \tag{10h}$$

$$\overset{*}{y}{}^{(m)}_{3r} = (1 - \delta_{m0})\tilde{b}_{1r} \quad , \quad (r = 1, 2, 3) \tag{10i}$$

where

$$\tilde{c}_{80} = -(1 - \nu^2)\left(\frac{3}{8} + 3\alpha^2 + \alpha^4\right) \tag{11a}$$

$$\tilde{c}_{81} = -(1 - \nu^2)(3\alpha + 4\alpha^3) \tag{11b}$$

$$\tilde{c}_{82} = -(1 - \nu^2)\left(\frac{1}{2} + 3\alpha^2\right) \tag{11c}$$

$$\tilde{c}_{83} = -(1 - \nu^2)\alpha \tag{11d}$$

$$\tilde{c}_{84} = -(1 - \nu^2)\frac{1}{8} \tag{11e}$$

$$\hat{b}_{10} = (1 - \nu^2)\left(\frac{3}{2}\alpha + \alpha^3\right) \tag{11f}$$

$$\tilde{b}_{11} = (1 - \nu^2)\left(\frac{3}{4} + 3\alpha^2\right) \tag{11g}$$

$$\tilde{b}_{12} = (1 - \nu^2)\left(\frac{3}{2}\alpha\right) \tag{11h}$$

$$\tilde{b}_{13} = (1 - \nu^2)\frac{1}{4} \tag{11i}$$

$$\tilde{a}_{40} = (1 - \nu^2)\left(\frac{1}{2} + \alpha^2\right) \tag{11j}$$

$$\tilde{a}_{41} = (1 - \nu^2)\, 2\alpha \tag{11k}$$

$$\tilde{a}_{42} = (1 - \nu^2)\, \frac{1}{2} \tag{11ℓ}$$

In Eqs. (10) and (11), we have used the notations given by Eqs. (VI – 6), (VI – 25), (VI – 26), and (VI – 27).

By letting m take on the values m = 0, 1, 2, ..., in Eqs. (6), we obtain an infinite system of algebraic equations in which the unknowns are the Fourier coefficients U_m, V_m, and W_m. The coefficients in this system of equations are shown in Table VI-1. Using matrix notation, we rewrite Eqs. (6) as

$$[R]\{V\} - \left(\frac{1}{\Omega}\right)[S - \lambda T]\{V\} = \{0\} \tag{12}$$

where $[R]$ and $[S - \lambda T]$ are square matrices formed by the coefficients $\left[\overset{*}{x}{}_{ij}^{(m)}, \overset{*}{y}{}_{ij}^{(m)}, \overset{*}{z}{}_{ij}^{(m)}\right]$ and $\left[\bar{x}_{ij}^{(m)} - \lambda\hat{x}_{ij}^{(m)}, \bar{y}_{ij}^{(m)} - \lambda\hat{y}_{ij}^{(m)}, \bar{z}_{ij}^{(m)} - \lambda\hat{z}_{ij}^{(m)}\right]$, respectively; $\{V\}$ is a column vector formed by the unknown Fourier coefficients U_m, V_m, and W_m. The elements of the $[R]$, $[S]$, and $[T]$ matrices can be obtained from Table VI-1 and Eqs. (7), (10), (11), and (VI – 15).

3.2 Mode B

For the vibration mode which is antimetric about the plane $\psi = 0, \pi$ (plane A-A in Fig. VI-1), the displacement components are represented by the Fourier series:

$$u_n(\psi) = \sum_{m=1}^{\infty} \tilde{U}_m \sin m\psi = \sum_{m=1}^{\infty} \tilde{U}_m S_m \tag{13a}$$

$$v_n(\psi) = \sum_{m=0}^{\infty} \tilde{V}_m \cos m\psi = \sum_{m=0}^{\infty} \tilde{V}_m C_m \tag{13b}$$

$$w_n(\psi) = \sum_{m=1}^{\infty} \tilde{W}_m \sin m\psi = \sum_{m=1}^{\infty} \tilde{W}_m S_m \tag{13c}$$

The vibration equations for Mode B are

$$\tilde{z}_{10}^{(m)} \tilde{U}_m + \sum_{r=1}^{3} \tilde{z}_{2r}^{(m)} \tilde{U}_{|m-r|} + \sum_{r=1}^{3} \tilde{z}_{3r}^{(m)} \tilde{U}_{m+r}$$

$$+ \tilde{z}_{40}^{(m)} \tilde{V}_m + \sum_{r=1}^{4} \tilde{z}_{5r}^{(m)} \tilde{V}_{|m-r|} + \sum_{r=1}^{4} \tilde{z}_{6r}^{(m)} \tilde{V}_{m+r}$$

$$+ \tilde{z}_{70}^{(m)} \tilde{W}_m + \sum_{r=1}^{4} \tilde{z}_{8r}^{(m)} \tilde{W}_{|m-r|} + \sum_{r=1}^{4} \tilde{z}_{9r}^{(m)} \tilde{W}_{m+r} = 0, m = (1, 2, \ldots) \tag{14a}$$

$$\tilde{x}_{10}^{(m)} \tilde{U}_m + \tilde{x}_{21}^{(m)} \tilde{U}_{|m-1|} + \tilde{x}_{31}^{(m)} \tilde{U}_{m+1} + \tilde{x}_{40}^{(m)} \tilde{V}_m + \sum_{r=1}^{2} \tilde{x}_{5r}^{(m)} \tilde{V}_{|m-r|} + \sum_{r=1}^{2} \tilde{x}_{6r}^{(m)} \tilde{V}_{m+r}$$

$$+ \tilde{x}_{70}^{(m)} \tilde{W}_m + \sum_{r=1}^{2} \tilde{x}_{8r}^{(m)} \tilde{W}_{|m-r|} + \sum_{r=1}^{2} \tilde{x}_{9r}^{(m)} \tilde{W}_{m+r} = 0, m = (0, 1, 2, \ldots) \tag{14b}$$

$$\tilde{y}_{10}^{(m)} \tilde{U}_m + \sum_{r=1}^{3} \tilde{y}_{2r}^{(m)} \tilde{U}_{|m-r|} + \sum_{r=1}^{3} \tilde{y}_{3r}^{(m)} \tilde{U}_{m+r} + \tilde{y}_{40}^{(m)} \tilde{V}_m + \sum_{r=1}^{2} \tilde{y}_{5r}^{(m)} \tilde{V}_{|m-r|}$$

$$+ \sum_{r=1}^{2} \tilde{y}_{6r}^{(m)} \tilde{V}_{m+r} + \tilde{y}_{70}^{(m)} \tilde{W}_m + \sum_{r=1}^{2} \tilde{y}_{8r}^{(m)} \tilde{W}_{|m-r|} + \sum_{r=1}^{2} \tilde{y}_{9r}^{(m)} \tilde{W}_{m+r} = 0$$

$$m = (1, 2, \ldots) \tag{14c}$$

where

$$\tilde{z}_{ij}^{(m)} = \overset{*}{\zeta}{}_{ij}^{(m)} - \left(\frac{1}{\Omega}\right)\left[\bar{\zeta}_{ij}^{(m)} - \lambda\hat{\zeta}_{ij}^{(m)}\right] \tag{15a}$$

$$\tilde{x}_{ij}^{(m)} = \overset{*}{\xi}{}_{ij}^{(m)} - \left(\frac{1}{\Omega}\right)\left[\bar{\xi}_{ij}^{(m)} - \lambda\hat{\xi}_{ij}^{(m)}\right] \tag{15b}$$

$$\tilde{y}_{ij}^{(m)} = \overset{*}{\eta}{}_{ij}^{(m)} - \left(\frac{1}{\Omega}\right)\left[\bar{\eta}_{ij}^{(m)} - \lambda\hat{\eta}_{ij}^{(m)}\right] \tag{15c}$$

The coefficients $\bar{\zeta}_{ij}^{(m)}$, $\bar{\xi}_{ij}^{(m)}$, $\bar{\eta}_{ij}^{(m)}$, $\hat{\zeta}_{ij}^{(m)}$, $\hat{\xi}_{ij}^{(m)}$, and $\hat{\eta}_{ij}^{(m)}$ in Eqs. (15) are given by Eqs. (VI – 38). The remaining nonzero coefficients in Eqs. (15) are:

$$\overset{*}{\zeta}{}_{70}^{(m)} = 2\,\tilde{c}_{80} \tag{16a}$$

$$\overset{*}{\zeta}{}_{8r}^{(m)} = -\epsilon_{mr}\tilde{c}_{8r} \quad , \quad (r = 1, 2, 3, 4) \tag{16b}$$

$$\overset{*}{\zeta}{}_{9r}^{(m)} = \tilde{c}_{8r} \quad , \quad (r = 1, 2, 3, 4) \tag{16c}$$

$$\overset{*}{\xi}{}_{40}^{(m)} = 2\,\tilde{a}_{40} \tag{16d}$$

$$\overset{*}{\xi}{}_{5r}^{(m)} = (1 + \delta_{mr})\tilde{a}_{4r} \quad , \quad (r = 1, 2) \tag{16e}$$

$$\overset{*}{\xi}{}_{6r}^{(m)} = (1 - \delta_{m0})\tilde{a}_{4r} \quad , \quad (r = 1, 2) \tag{16f}$$

$$\overset{*}{\eta}{}_{10}^{(m)} = 2\,\tilde{b}_{10} \tag{16g}$$

$$\overset{*}{\eta}{}_{2r}^{(m)} = -\epsilon_{mr}\tilde{b}_{1r} \quad , \quad (r = 1, 2, 3) \tag{16h}$$

$$\overset{*}{\eta}{}_{3r}^{(m)} = \tilde{b}_{1r} \quad , \quad (r = 1, 2, 3) \tag{16i}$$

Using matrix notation, we rewrite Eqs. (14) as:

$$[\tilde{R}]\{\tilde{V}\} - \left(\frac{1}{\Omega}\right)[\tilde{S} - \lambda\tilde{T}]\{\tilde{V}\} = \{0\} \tag{17}$$

where $[\tilde{R}]$ and $[\tilde{S} - \lambda\tilde{T}]$ are square matrices formed by the coefficients $\left[\overset{*}{\xi}{}^{(m)}_{ij}, \overset{*}{\eta}{}^{(m)}_{ij}, \overset{*}{\zeta}{}^{(m)}_{ij}\right]$ and $\left[\bar{\xi}^{(m)}_{ij} - \lambda\hat{\xi}^{(m)}_{ij}, \bar{\eta}^{(m)}_{ij} - \lambda\hat{\eta}^{(m)}_{ij}, \bar{\zeta}^{(m)}_{ij} - \lambda\hat{\zeta}^{(m)}_{ij}\right]$, respectively; $\{\tilde{V}\}$ is a column vector formed by the unknown Fourier coefficients $\tilde{U}_m$, $\tilde{V}_m$, and $\tilde{W}_m$. The elements of the $[\tilde{R}]$, $[\tilde{S}]$, and $[\tilde{T}]$ matrices can be obtained from Table VI-2 and Eqs. (11), (15), (16), and (VI – 15).

5.3 Eigenvalues and Eigenfunctions

The eigenvalues and corresponding eigenfunctions for Modes A and B can be obtained from Eq. (12) and Eq. (17), respectively. For a given value of the pressure parameter λ, the lowest eigenvalue Ω for which these equations admit a nontrivial solution may be obtained by the same method as was used in Chapter VII for the stability analysis of a toroidal shell. The corresponding eigenfunctions or mode shapes can be determined after substitution of the computed eigenvalue in Eqs. (12) and (17).

IX

CONCLUDING REMARKS

This work presents a theoretical investigation of the stability of thin shells of revolution. Stability equations are derived for a shell of revolution under general loading conditions. These equations are specialized for a toroidal shell loaded by uniform external pressure. The resulting equations are solved by use of series expansions in the circumferential and meridional directions for the displacement components that develop during buckling. The analysis shows that a toroidal shell can buckle in a mode which is either symmetric or antimetric about the equatorial plane and that the corresponding buckling pressures are always close to each other. Axially symmetric as well as asymmetric buckling modes are considered. The numerical results show that the asymmetric modes give lower buckling pressures. Design curves which give nondimensional buckling pressures for a wide range of the toroidal shell's geometric parameters are presented. In addition, the variation of the mode shapes with the geometric parameters is illustrated. In a comparison between the results of the present theory and the few available tests on toroidal shells, it is shown that test and theory agree to within 10%. For the limiting case of a sphere under external pressure, the well-known classical result ($p = 1.21\ Eh^2/a^2$) is obtained numerically for both the asymmetric and axially symmetric buckling modes. Finally, equations governing the free vibrations of a prestressed shell of revolution are presented and specialized for a toroidal shell subject to external or internal pressure.

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